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Stochastic Comparisons of Lifetimes of Two Series and Parallel Systems
with Location-Scale Family Distributed Components having
Archimedean Copulas

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Stochastic Comparisons of Lifetimes of Two Series and Parallel Systems with Location-Scale Family Distributed Components having Archimedean Copulas

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Abstract- In this paper, we compare the lifetimes of two series and two parallel systems stochastically where the lifetime of each component follows location-scale (LS) family of distributions. The comparison is carried out under two scenarios: one, that the components of the systems have a dependent structure sharing Archimedean copula and two, that the components are independently distributed. It is shown that the systems with components in series or parallel sharing Archimedean copula with more dispersion in the location or scale parameters results in better performance in the sense of the usual stochastic order. It is also shown that if the components are independently distributed, it is possible to obtain more generalized results as compared to the dependent set-up. The results in this paper generalizes similar results in both independent and dependent set up for exponential and Weibull distributed components.

Index Terms- Location-scale family of distributions; Series system; Parallel system; Archimedean copula; stochastic order; likelihood ratio order; ageing faster order; multiple outlier model; majorization

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<div align="center">**NOTATION**</div>

t	A fixed time point
rv	random variable
iid	independently and identically distributed
df	distribution function
LS family	Location-scale family of distributions
$F_X(\cdot)$	cumulative distribution function of random variable X
$\bar{F}_X(\cdot)$	survival (reliability) function of random variable X
$r_X(\cdot)$	hazard rate function of random variable X
$\tilde{r}_X(\cdot)$	reversed hazard rate function of random variable X
$X_{1:n}$	minimum order statistic (series system)
$X_{n:n}$	maximum order statistic (parallel system)
$X \leq_{st} Y$	X is said to be smaller than Y in usual stochastic order
$X \leq_{hr} Y$	X is said to be smaller than Y in hazard rate order
$X \leq_{rh} Y$	X is said to be smaller than Y in reversed hazard rate order
$X \leq_{R-hr} Y$	X is said to be smaller than Y in ageing faster order in terms of the hazard rate order
$X \leq_{lr} Y$	X is said to be smaller than Y in likelihood ratio order
ψ	generator of Archimedean copula
ϕ	inverse of ψ
$s - convex$	Schur-convex
$s - concave$	Schur-concave
$\mathbf{x} \overset{m}{\succeq} \mathbf{y}$	$\mathbf{x}$ is said to majorize $\mathbf{y}$
$\mathbf{x} \overset{w}{\succeq} \mathbf{y}$	$\mathbf{x}$ is said to weakly supermajorize $\mathbf{y}$
$\mathbf{x} \succeq_w \mathbf{y}$	$\mathbf{x}$ is said to weakly submajorize $\mathbf{y}$
$\mathbf{x} \overset{p}{\succeq} \mathbf{y}$	$\mathbf{x}$ is said to be p-larger than $\mathbf{y}$
$\mathbf{x} \overset{rm}{\succeq} \mathbf{y}$	$\mathbf{x}$ is said to reciprocally majorize $\mathbf{y}$
$a \overset{sign}{=} b$	a and b have the same sign
$k'(t)$	first derivative of $k(t)$ with respect to t
$k''(t)$	second derivative of $k(t)$ with respect to t

1   INTRODUCTION

Stochastic comparison of system lifetimes has always been a relevant topic in reliability optimization and life testing experiments. These comparisons can be used to choose the best system structure under different criteria or to study where to place the different components in a system structure. If $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ denote the order statistics corresponding to the random variables $X_1, X_2, \ldots, X_n$, then the lifetime of a series and parallel system correspond to the smallest $(X_{1:n})$, and the largest $(X_{n:n})$ order statistic respectively. Classical theory of

systems assumes that the lifetimes of the components are *iid* (see David and Nagaraja [1]). Considerable amount of work has also been carried out in the past years in comparing the lifetimes of heterogeneous independent components of systems (largely on the smallest and the largest order statistics) with certain underlying distributions on both finite and infinite range with respect to usual stochastic ordering, hazard rate ordering, reversed hazard rate ordering and likelihood ratio ordering. One may refer to Dykstra *et al.* [2], Zhao and Balakrishnan ([3]), Balakrishnan *et al.* [4], Torrado and Kochar [5], Torrado [6], Fang and Balakrishnan [7], Kundu *et al.* [8], Kundu and Chowdhury ([9],[10]), Chowdhury and Kundu [11] and Hazra *et al.* [12] for more detail. There are few work in the same area where the authors have compared systems stochastically through relative ageing, also known as ageing faster ordering in terms of hazard rate or reversed hazard rate ordering. One may refer to Sengupta and Deshpande [13], Rezaei *et al.* [14] and Li and Li [15] for more detail.

However, in practical situations, the components of a system may have a structural dependence which result in a set of statistically dependent observations. The dependence structure of the components are investigated by researchers very recently with the help of copulas. Navarro and Spizzichino [16] studied stochastic orders of series and parallel systems with components sharing a common copula. Rezapour and Alamatsaz [17] investigated stochastic orders on order statistics from samples with different survival Archimedean copulas. Li and Li [18] studied stochastic ordering of the sample minimums of Weibull samples sharing a common Archimedean survival copula. Li and Fang [19] compared the lifetimes of parallel systems with proportional hazard rate (PHR) components following the Archimedean copula which was further investigated by Li *et al.* [20] and Fang *et al.* [21].

The location-scale family of distributions is commonly used in lifetime studies. The most widely used statistical distributions are either members of this class or closely related to this class of distributions; such as exponential, normal, Weibull, lognormal, loglogistic, logistic, and extreme value distributions. Methods of inference and statistical theory for the general family can be applied to this large, important class of models. A random variable X is said to follow location-scale family distribution, written as $LS(\lambda, \sigma, F)$ and will be termed as LS family hereafter, if the distribution function of X is given by
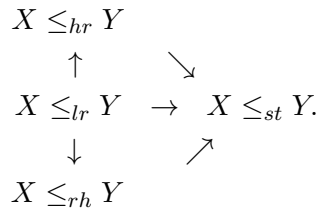
$$F(x; \lambda, \sigma, F) = F\left(\frac{x - \lambda}{\sigma}\right), \quad x > \lambda, \ \sigma > 0 \ x, \ \lambda, \ \sigma \in \mathbb{R}, \tag{1}$$

where λ and σ are the location and the scale parameter respectively and $F(\cdot)$ is the baseline distribution function of the rv X. Although significant previous research has compared series or parallel systems of heterogenous independent components including scale family of distributions (see Khaledi *et al.* [22], Li *et al.* [20], Kochar and Torrado [23] and Li and Li [15]), there has been few work examining similar comparisons for dependent components; furthermore, all such comparisons for dependent components assume either PHR components or Weibull distributed components. As LS family of distributions cover a large pool of lifetime distributions, we are motivated to assume the component lifetimes to follow LS family and to compare the

lifetimes of two series or parallel systems stochastically, assuming a dependence structure in the components. In this sense, the paper distinguishes itself from the other few existing work. It generalizes the results on stochastic comparison of lifetimes of two series or parallel systems with heterogeneous dependent and independent components. Moreover, the comparisons are carried out under different baseline distributions of LS family with stochastic ordering, hazard rate orderings and reversed hazard rate orderings between them. The rest of the paper is organized as follows. In Section 2, we have given the required definitions and some useful lemmas which are used throughout the paper. Results related to stochastic comparison of series systems with heterogeneous independent and dependent components are derived in Section 3. Section 4 discusses some results on parallel systems with dependent components.

2   PRELIMINARIES

For two absolutely continuous random variables X and Y with distribution functions $F(\cdot)$ and $G(\cdot)$, survival functions $\overline{F}(\cdot)$ and $\overline{G}(\cdot)$, density functions $f(\cdot)$ and $g(\cdot)$, hazard rate functions $r(\cdot)$ and $s(\cdot)$ and reversed hazard rate functions $\tilde{r}(\cdot)$ and $\tilde{s}(\cdot)$ respectively, X is said to be smaller than Y in i) *likelihood ratio order* (denoted as $X \leq_{lr} Y$), if, for all t, $\frac{g(t)}{f(t)}$ increases in t, ii) *hazard rate order* (denoted as $X \leq_{hr} Y$), if, for all t, $\frac{\overline{G}(t)}{\overline{F}(t)}$ increases in t or equivalently $r(t) \geq s(t)$, iii) *reversed hazard rate order* (denoted as $X \leq_{rh} Y$), if, for all t, $\frac{G(t)}{F(t)}$ increases in t or equivalently $\tilde{r}(t) \leq \tilde{s}(t)$, iii) *Ageing faster order in terms of the hazard rate order* (denoted as $X \leq_{R-hr} Y$), if, for all t, $\frac{r_X(t)}{r_Y(t)}$ increases in t, and iv) *usual stochastic order* (denoted as $X \leq_{st} Y$), if $F(t) \geq G(t)$ for all t. In the following diagram we present a chain of implications of the stochastic orders. For more on stochastic orders, see Shaked and Shanthikumar [26].

$$
\begin{array}{ccc}
X \leq_{hr} Y & & \\
\uparrow & \searrow & \\
X \leq_{lr} Y & \rightarrow & X \leq_{st} Y. \\
\downarrow & \nearrow & \\
X \leq_{rh} Y & &
\end{array}
$$

The notion of majorization (Marshall *et al.* [24]) is essential for the understanding of the stochastic inequalities for comparing order statistics. Let $\mathbb{R}^n$ be an n-dimensional Euclidean space. Further, for any two real vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, write $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ and $y_{(1)} \leq y_{(2)} \leq \cdots \leq y_{(n)}$ as the increasing arrangements of the components of the vectors $\mathbf{x}$ and $\mathbf{y}$ respectively. The following definitions may be found in Marshall *et al.* [24].

Definition 1   i) *The vector $\mathbf{x}$ is said to majorize the vector $\mathbf{y}$ (written as $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$) if*

$$
\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}, \ j = 1, 2, \ldots, n-1, \ \ and \ \ \sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}.
$$

ii) The vector $\mathbf{x}$ is said to weakly supermajorize the vector $\mathbf{y}$ (written as $\mathbf{x} \overset{\mathrm{w}}{\succeq} \mathbf{y}$) if

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)} \quad \text{for } j = 1, 2, \ldots, n.$$

iii) The vector $\mathbf{x}$ is said to weakly submajorize the vector $\mathbf{y}$ (written as $\mathbf{x} \succeq_{\mathrm{w}} \mathbf{y}$) if

$$\sum_{i=j}^{n} x_{(i)} \geq \sum_{i=j}^{n} y_{(i)} \quad \text{for } j = 1, 2, \ldots, n.$$

(iv) The vector $\mathbf{x}$ is said to be p-larger than the vector $\mathbf{y}$ (written as $\mathbf{x} \overset{\mathrm{p}}{\succeq} \mathbf{y}$) if

$$\prod_{i=1}^{j} x_{(i)} \leq \prod_{i=1}^{j} y_{(i)} \quad \text{for } j = 1, 2, \ldots, n.$$

(v) The vector $\mathbf{x}$ is said to reciprocally majorize the vector $\mathbf{y}$ (written as $\mathbf{x} \overset{\mathrm{rm}}{\succeq} \mathbf{y}$) if

$$\sum_{i=1}^{j} \frac{1}{x_{(i)}} \geq \sum_{i=1}^{j} \frac{1}{y_{(i)}} \quad \text{for } j = 1, 2, \ldots, n.$$

It is not so difficult to show that $\mathbf{x} \overset{m}{\succeq} \mathbf{y} \Rightarrow \mathbf{x} \overset{w}{\succeq} \mathbf{y} \Rightarrow \mathbf{x} \overset{p}{\succeq} \mathbf{y} \Rightarrow \mathbf{x} \overset{rm}{\succeq} \mathbf{y}$.

Definition 2 *A function $\psi : \mathbb{R}^n \to \mathbb{R}$ is said to be Schur-convex (resp. Schur-concave) on $\mathbb{R}^n$ if*

$$\mathbf{x} \overset{m}{\succeq} \mathbf{y} \text{ implies } \psi(\mathbf{x}) \geq (resp. \ \leq) \ \psi(\mathbf{y}) \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

Definition 3 *For any integer r, a function $\psi : \mathbb{R} \to \mathbb{R}$ is said to be r-convex (resp. r-concave) on $\mathbb{R}$ if $\frac{d^r \psi(x)}{dx^r} \geq (\leq)0$ for all $x \in \mathbb{R}$.*

Notation 1 *Let us introduce the following notations.*

(i) $\mathcal{D}_+ = \{(x_1, x_2, \ldots, x_n) : x_1 \geq x_2 \geq \ldots \geq x_n > 0\}$.

(ii) $\mathcal{E}_+ = \{(x_1, x_2, \ldots, x_n) : 0 < x_1 \leq x_2 \leq \ldots \leq x_n\}$.

Let us first introduce the following lemmas which will be used in the next sections to prove the results.

Lemma 1 (*Lemma 3.1 of Kundu* et al. *[8]*) *Let $\varphi : \mathcal{D}_+ \to \mathbb{R}$ be a function, continuously differentiable on the interior of $\mathcal{D}_+$. Then, for $\mathbf{x}, \mathbf{y} \in \mathcal{D}_+$,*

$$\mathbf{x} \overset{m}{\succeq} \mathbf{y} \text{ implies } \varphi(\mathbf{x}) \geq (resp. \ \leq) \ \varphi(\mathbf{y})$$

if, and only if,

$$\varphi_{(k)}(\mathbf{z}) \text{ is decreasing (resp. increasing) in } k = 1, 2, \ldots, n,$$

where $\varphi_{(k)}(\mathbf{z}) = \partial \varphi(\mathbf{z})/\partial z_k$ denotes the partial derivative of φ with respect to its kth argument.

Lemma 2 (*Lemma 3.3 of Kundu* et al. *[8]*) *Let $\varphi : \mathcal{E}_+ \to \mathbb{R}$ be a function, continuously differentiable on the interior of $\mathcal{E}_+$. Then, for $\mathbf{x}, \mathbf{y} \in \mathcal{E}_+$,*

$$\mathbf{x} \overset{m}{\succeq} \mathbf{y} \text{ implies } \varphi(\mathbf{x}) \geq (\text{resp. } \leq) \varphi(\mathbf{y})$$

if, and only if,

$$\varphi_{(k)}(\mathbf{z}) \text{ is increasing (resp. decreasing) in } k = 1, 2, \ldots, n,$$

where $\varphi_{(k)}(\mathbf{z}) = \partial \varphi(\mathbf{z}) / \partial z_k$ denotes the partial derivative of φ with respect to its kth argument.

Lemma 3 (*Lemma 3.1 of Khaledi and Kochar [25]*) *Let $S \subseteq \mathbb{R}_+^n$. Further, let $\psi : S \to \mathbb{R}$ be a function. Then, for $\mathbf{x}, \mathbf{y} \in S$,*

$$\mathbf{x} \overset{p}{\succeq} \mathbf{y} \text{ implies } \psi(\mathbf{x}) \geq (\text{resp. } \leq) \psi(\mathbf{y})$$

if, and only if,

  (i) $\psi(e^{a_1}, \ldots, e^{a_n})$ is Schur-convex (resp. Schur-concave) in $(a_1, \ldots, a_n) \in S$,

  (ii) $\psi(e^{a_1}, \ldots, e^{a_n})$ is decreasing (resp. increasing) in a_i, for $i = 1, \ldots, n$,

where $a_i = \ln x_i$, for $i = 1, \ldots, n$.

Lemma 4 (*Lemma 4.1 of Hazra* et al. *[12]*) *Let $S \subseteq \mathbb{R}_+^n$. Further, let $\psi : S \to \mathbb{R}$ be a function. Then, for $\mathbf{x}, \mathbf{y} \in S$,*

$$\mathbf{x} \overset{rm}{\succeq} \mathbf{y} \text{ implies } \psi(\mathbf{x}) \geq (\text{resp. } \leq) \psi(\mathbf{y})$$

if, and only if,

  (i) $\psi(\frac{1}{a_1}, \ldots, \frac{1}{a_n})$ is Schur-convex (resp. Schur-concave) in $(a_1, \ldots, a_n) \in S$,

  (ii) $\psi(\frac{1}{a_1}, \ldots, \frac{1}{a_n})$ is increasing (resp. decreasing) in a_i, for $i = 1, \ldots, n$,

where $a_i = \frac{1}{x_i}$, for $i = 1, \ldots, n$.

Lemma 5 (*Theorem A.8 of Marshall* et al. *[24] p.p. 87*) *Let $S \subseteq \mathbb{R}^n$. Further, let $\varphi : S \to \mathbb{R}$ be a function. Then for $\mathbf{x}, \mathbf{y} \in S$,*

$$\mathbf{x} \succeq_w \mathbf{y} \Longrightarrow \varphi(\mathbf{x}) \geq (\text{resp. } \leq) \varphi(\mathbf{y})$$

if, and if, φ is both increasing (resp. decreasing) and Schur-convex (resp. Schur-concave) on S. Similarly,

$$\mathbf{x} \overset{w}{\succeq} \mathbf{y} \Longrightarrow \varphi(\mathbf{x}) \geq (\text{resp. } \leq) \varphi(\mathbf{y})$$

if, and if, φ is both decreasing (resp. increasing) and Schur-convex (resp. Schur-concave) on S.

Now, let us recall that a copula associated with a multivariate distribution function F is a function $C : [0,1]^n \longmapsto [0,1]$ satisfying: $F(x) = C(F_1(X_1), ..., F_n(X_n))$, where the F_i's, $1 \leq i \leq n$ are the univariate marginal distribution functions of X_is. Similarly, a survival copula associated with a multivariate survival function $\overline{F}$ is a function $\overline{C} : [0,1]^n \longmapsto [0,1]$ satisfying:

$$\overline{F}(x) = P(X_1 > x_1, ..., X_n > x_n) = \overline{C}\left(\overline{F}_1(x_1), ..., \overline{F}_n(x_n)\right),$$

where, for $1 \leq i \leq n$, $\overline{F}_i(\cdot) = 1 - F_i(\cdot)$ are the univariate survival functions. In particular, a copula C is Archimedean if there exists a generator $\psi : [0,\infty] \longmapsto [0,1]$ such that

$$C(\mathbf{u}) = \psi\left(\psi^{-1}(u_1), ..., \psi^{-1}(u_d)\right).$$

For C to be Archimedean copula, it is sufficient and necessary that ψ satisfies $i)$ $\psi(0) = 1$ and $\psi(\infty) = 0$ and $ii)$ ψ is $d-$monotone, i.e. $\frac{(-1)^k d^k \psi(s)}{ds^k} \geq 0$ for $k \in \{0, 1, ..., d-2\}$ and $\frac{(-1)^{d-2} d^{d-2} \psi(s)}{ds^{d-2}}$ is decreasing and convex. Archimedean copulas cover a wide range of dependence structures including the independence copula and the Clayton copula. For more detail on Archimedean copula, see, Nelsen [27] and McNeil and Nĕslehová [28]. In this paper, Archimedean copula is specifically employed to model on the dependence structure among random variables in a sample. The following important lemma is used in the next sections to prove some of the important theorems.

Lemma 6 (*Li and Fang [19]*) *For two n-dimensional Archimedean copulas $C_{\psi_1}(\mathbf{u})$ and $C_{\psi_2}(\mathbf{u})$, with $\phi_2 = \psi_2^{-1} = \sup\{x \in \mathbb{R} : \psi(x) > u\}$, the right continuous inverse, if $\phi_2 \circ \psi_1$ is super-additive, then $C_{\psi_1}(\mathbf{u}) \leq C_{\psi_2}(\mathbf{u})$ for all $\mathbf{u} \in [0,1]^n$. Recall that a function f is said to be super-additive if $f(x+y) \geq f(x) + f(y)$, for all x and y in the domain of f.*

3 COMPARISON OF SERIES SYSTEMS WITH LS DISTRIBUTED COMPONENTS

This section is devoted to the comparison of two series systems with heterogenous LS family distributed components. The comparison is carried out under two scenarios: one, that the components have a dependent structure sharing Archimedean copulas and the other is that the components are independently distributed.

3.1 Some Results on Heterogenous Dependent Components

Let, X and Y be two random variables having distribution functions $F(\cdot)$ and $G(\cdot)$ respectively. Also suppose that $X_i \sim LS(\lambda_i, \sigma_i, F)$ and $Y_i \sim LS(\mu_i, \xi_i, G)$ $(i = 1, 2, \ldots, n)$ be two sets of n dependent random variables with Archimedean copulas having generators ψ_1 $\left(\phi_1 = \psi_1^{-1}\right)$ and ψ_2 $\left(\phi_2 = \psi_2^{-1}\right)$ respectively. Also suppose that $\overline{G}_{1:n}(\cdot)$ and $\overline{H}_{1:n}(\cdot)$ be the survival functions of $X_{1:n}$ and $Y_{1:n}$ respectively. Then,

$$\overline{G}_{1:n}(t) = \psi_1\left[\sum_{k=1}^n \phi_1\left\{1 - F\left(\frac{t - \lambda_k}{\sigma_k}\right)\right\}\right], \quad t > max(\lambda_k, \ \forall k),$$

and

$$\overline{H}_{1:n}(t) = \psi_2 \left[\sum_{k=1}^{n} \phi_2 \left\{ 1 - G \left(\frac{t - \mu_k}{\xi_k} \right) \right\} \right], \ t > max(\mu_k, \ \forall k).$$

Let $r_X(u)$ and $r_Y(u)$ are the hazard rate functions of the random variables X and Y respectively. The first two theorems show that usual stochastic ordering exists between $X_{1:n}$ and $Y_{1:n}$ under weak majorization order of the scale parameters and stochastic ordering between X and Y.

Theorem 1 *Let $X_1, X_2, ..., X_n$ be a set of dependent random variables sharing Archimedean copula having generator ψ_1 such that $X_i \sim LS(\lambda_i, \sigma_i, F)$, $i = 1, 2, ..., n$. Let $Y_1, Y_2, ..., Y_n$ be another set of dependent random variables sharing Archimedean copula having generator ψ_2 such that $Y_i \sim LS(\lambda_i, \xi_i, G)$, $i = 1, 2, ..., n$. Assume that $\boldsymbol{\sigma}, \boldsymbol{\xi}$ and $\boldsymbol{\lambda} \in \mathcal{D}_+$ (or $\mathcal{E}_+$). Further suppose that $\phi_2 \circ \psi_1$ is super-additive, ψ_1 or ψ_2 is log-concave and $X \leq_{st} Y$. If either $r_X(u)$ or $r_Y(u)$ is decreasing in u, then $\frac{1}{\boldsymbol{\sigma}} \overset{w}{\preceq} \frac{1}{\boldsymbol{\xi}}$ implies $X_{1:n} \leq_{st} Y_{1:n}$.*

Proof: By Lemma 6, super-additivity of $\phi_2 \circ \psi_1$ implies that

$$\psi_1 \left[\sum_{k=1}^{n} \phi_1 \left\{ 1 - F \left(\frac{t - \lambda_k}{\sigma_k} \right) \right\} \right] \leq \psi_2 \left[\sum_{k=1}^{n} \phi_2 \left\{ 1 - F \left(\frac{t - \lambda_k}{\sigma_k} \right) \right\} \right]. \tag{2}$$

As $X \leq_{st} Y$ and ϕ_2 and ψ_2 are decreasing in x, it can be easily shown that

$$\psi_2 \left[\sum_{k=1}^{n} \phi_2 \left\{ 1 - F \left(\frac{t - \lambda_k}{\sigma_k} \right) \right\} \right] \leq \psi_2 \left[\sum_{k=1}^{n} \phi_2 \left\{ 1 - G \left(\frac{t - \lambda_k}{\sigma_k} \right) \right\} \right]. \tag{3}$$

(2) and (3) together implies that

$$\psi_1 \left[\sum_{k=1}^{n} \phi_1 \left\{ 1 - F \left(\frac{t - \lambda_k}{\sigma_k} \right) \right\} \right] \leq \psi_2 \left[\sum_{k=1}^{n} \phi_2 \left\{ 1 - G \left(\frac{t - \lambda_k}{\sigma_k} \right) \right\} \right].$$

Therefore, to prove the result it suffices to prove that

$$\psi_2 \left[\sum_{k=1}^{n} \phi_2 \left\{ 1 - G \left(\frac{t - \lambda_k}{\sigma_k} \right) \right\} \right] \leq \psi_2 \left[\sum_{k=1}^{n} \phi_2 \left\{ 1 - G \left(\frac{t - \lambda_k}{\xi_k} \right) \right\} \right].$$

Let us assume that

$$\psi_2 \left[\sum_{k=1}^{n} \phi_2 \left\{ 1 - G \left(\frac{t - \lambda_k}{\sigma_k} \right) \right\} \right] = \psi_2 \left[\sum_{k=1}^{n} \phi_2 \left\{ 1 - G \left(p_k(t - \lambda_k) \right) \right\} \right] = \Psi(\boldsymbol{p}),$$

where $\boldsymbol{p} = (p_1, p_2, ..., p_n) = \left(\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, ..., \frac{1}{\sigma_n} \right)$. Hence, by Lemma 5, to prove the result, it suffices to prove that $\Psi(\boldsymbol{p})$ is decreasing and s-convex in $\boldsymbol{p}$. Now, if $\boldsymbol{\sigma}, \boldsymbol{\lambda} \in \mathcal{D}_+$ (or $\mathcal{E}_+$) and $r_Y(u)$ is decreasing in u, then, for all $i \leq j$,

$$r_Y(p_i(t - \lambda_i)) \geq (\leq) r_Y(p_j(t - \lambda_j)).$$

Again, as ψ_2 is log-concave, giving that $\frac{\psi_2(u)}{\psi_2'(u)}$ is increasing in u, then, for all $i \leq j$, it can be written that

$$- r_Y \left(p_i(t - \lambda_i) \right) \frac{\psi_2 \left(u_i \right)}{\psi_2' \left(u_i \right)} \geq (\leq) - r_Y \left(p_j(t - \lambda_j) \right) \frac{\psi_2 \left(u_j \right)}{\psi_2' \left(u_j \right)}, \tag{4}$$

where $u_i = \phi_2 \left[1 - G \left(p_i(t - \lambda_i) \right) \right]$. Now, differentiating $\Psi(\boldsymbol{p})$ with respect to p_i, we get

$$\frac{\partial \Psi}{\partial p_i} = -\psi_2' \left[\sum_{k=1}^{n} \phi_2 \left\{ 1 - G \left(p_k(t - \lambda_k) \right) \right\} \right] (t - \lambda_i) r_Y \left(p_i(t - \lambda_i) \right)$$
$$\frac{\psi_2 \left[\phi_2 \left\{ 1 - G \left(p_i(t - \lambda_i) \right) \right\} \right]}{\psi_2' \left[\phi_2 \left\{ 1 - G \left(p_i(t - \lambda_i) \right) \right\} \right]} \leq 0,$$

proving that $\Psi(\boldsymbol{p})$ is decreasing in each p_i. Moreover, using (4), it can be easily shown that $\frac{\partial \Psi}{\partial p_i} - \frac{\partial \Psi}{\partial p_j} \leq (\geq)0$. Thus, by Lemma 2 (Lemma 1) it can be written that $\Psi(\boldsymbol{p})$ is s-convex in $\boldsymbol{p}$. This proves the result. □

The next theorem discusses about the stochastic ordering between $X_{1:n}$ and $Y_{1:n}$ under weak majorization order of the scale parameters when ψ_1 or ψ_2 is log-convex. The theorem can be proved in the similar line as of the previous one and hence the proof is omitted.

Theorem 2 *Let $X_1, X_2, ..., X_n$ be a set of dependent random variables sharing Archimedean copula having generator ψ_1 such that $X_i \sim LS(\lambda_i, \sigma_i, F)$, $i = 1, 2, ..., n$. Let $Y_1, Y_2, ..., Y_n$ be another set of dependent random variables sharing Archimedean copula having generator ψ_2 such that $Y_i \sim LS(\lambda_i, \xi_i, G)$, $i = 1, 2, ..., n$. Assume that $\boldsymbol{\sigma}, \boldsymbol{\xi}$ and $\boldsymbol{\lambda} \in \mathcal{D}_+$ (or $\mathcal{E}_+$). Further suppose that $\phi_2 \circ \psi_1$ is super-additive, ψ_1 or ψ_2 is log-convex and $X \leq_{st} Y$. If either $r_X(u)$ or $r_Y(u)$ is increasing in u, then $\frac{1}{\boldsymbol{\sigma}} \succeq_w \frac{1}{\boldsymbol{\xi}}$ implies $X_{1:n} \leq_{st} Y_{1:n}$.*

In the next theorem usual stochastic ordering between $X_{1:n}$ and $Y_{1:n}$ has been established under p and rm orderings of the scale parameters.

Theorem 3 *Let $X_1, X_2, ..., X_n$ be a set of dependent random variables sharing Archimedean copula having generator ψ_1 such that $X_i \sim LS(\lambda_i, \sigma_i, F)$, $i = 1, 2, ..., n$. Let $Y_1, Y_2, ..., Y_n$ be another set of dependent random variables sharing Archimedean copula having generator ψ_2 such that $Y_i \sim LS(\lambda_i, \xi_i, G)$, $i = 1, 2, ..., n$. Assume that $\boldsymbol{\sigma}, \boldsymbol{\xi}, \boldsymbol{\lambda} \in \mathcal{D}_+$ (or $\mathcal{E}_+$). Further suppose that $\phi_2 \circ \psi_1$ is super-additive, ψ_1 or ψ_2 is log-concave and $X \leq_{st} Y$, then,*

i) $\frac{1}{\boldsymbol{\sigma}} \overset{p}{\preceq} \frac{1}{\boldsymbol{\xi}} \Rightarrow X_{1:n} \leq_{st} Y_{1:n}$, if $u r_X(u)$ or $u r_Y(u)$ is decreasing in u;

ii) $\frac{1}{\boldsymbol{\sigma}} \overset{rm}{\preceq} \frac{1}{\boldsymbol{\xi}} \Rightarrow X_{1:n} \leq_{st} Y_{1:n}$, if $u^2 r_X(u)$ or $u^2 r_Y(u)$ is decreasing in u.

Proof: i) As $\phi_2 \circ \psi_1$ is super-additive, by Lemma 6 it can be written that

$$\psi_1 \left[\sum_{k=1}^{n} \phi_1 \left\{ 1 - F \left(\frac{t - \lambda_k}{\sigma_k} \right) \right\} \right] \leq \psi_2 \left[\sum_{k=1}^{n} \phi_2 \left\{ 1 - F \left(\frac{t - \lambda_k}{\sigma_k} \right) \right\} \right]. \tag{5}$$

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Again noticing the fact that $X \leq_{st} Y$ and ϕ_2 and ψ_2 are decreasing function of x, it can be easily shown that

$$\psi_2 \left[\sum_{k=1}^{n} \phi_2 \left\{ 1 - F \left(\frac{t - \lambda_k}{\sigma_k} \right) \right\} \right] \leq \psi_2 \left[\sum_{k=1}^{n} \phi_2 \left\{ 1 - G \left(\frac{t - \lambda_k}{\sigma_k} \right) \right\} \right]. \qquad (6)$$

So, (5) and (6) together gives,

$$\psi_1 \left[\sum_{k=1}^{n} \phi_1 \left\{ 1 - F \left(\frac{t - \lambda_k}{\sigma_k} \right) \right\} \right] \leq \psi_2 \left[\sum_{k=1}^{n} \phi_2 \left\{ 1 - G \left(\frac{t - \lambda_k}{\sigma_k} \right) \right\} \right].$$

Therefore, to prove the result, it suffices to prove that

$$\psi_2 \left[\sum_{k=1}^{n} \phi_2 \left\{ 1 - G \left(\frac{t - \lambda_k}{\sigma_k} \right) \right\} \right] \leq \psi_2 \left[\sum_{k=1}^{n} \phi_2 \left\{ 1 - G \left(\frac{t - \lambda_k}{\xi_k} \right) \right\} \right].$$

Let us assume that
$\psi_2 \left[\sum_{k=1}^{n} \phi_2 \left\{ 1 - G \left(\frac{t - \lambda_k}{\sigma_k} \right) \right\} \right] = \psi_2 \left[\sum_{k=1}^{n} \phi_2 \left\{ 1 - G \left(e^{p_k} (t - \lambda_k) \right) \right\} \right] = \Psi(\boldsymbol{p})$, where $\boldsymbol{p} = (p_1, p_2, ..., p_n) = \left(\log \frac{1}{\sigma_1}, \log \frac{1}{\sigma_2}, ..., \log \frac{1}{\sigma_n} \right)$. Hence, by Lemma 3, it suffices to prove that $\Psi(\boldsymbol{p})$ is decreasing and s-convex in $\boldsymbol{p}$. Now, if $\boldsymbol{\lambda}, \boldsymbol{\sigma} \in \mathcal{D}_+$ (or $\mathcal{E}_+$) and $u r_Y(u)$ is decreasing in u, then for all $i \leq j$,

$$e^{p_i} (t - \lambda_i) r_Y \left(e^{p_i} (t - \lambda_i) \right) \geq (\leq) e^{p_j} (t - \lambda_j) r_Y \left(e^{p_j} (t - \lambda_j) \right).$$

Again, if ψ_2 is log-concave, then $\frac{\psi_2(u)}{\psi_2'(u)}$ is increasing in u, which implies that

$$- e^{p_i} (t - \lambda_i) r_Y \left(e^{p_i} (t - \lambda_i) \right) \frac{\psi_2(u_i)}{\psi_2'(u_i)} \geq (\leq) - e^{p_j} (t - \lambda_j) r_Y \left(e^{p_j} (t - \lambda_j) \right) \frac{\psi_2(u_j)}{\psi_2'(u_j)}; \ i \leq j, \qquad (7)$$

with $u_i = \phi_2 \left[1 - G \left(e^{p_i} (t - \lambda_i) \right) \right]$. Now, differentiating $\Psi(\boldsymbol{p})$ with respect to p_i, we get,

$$\frac{\partial \Psi}{\partial p_i} = -\psi_2' \left[\sum_{k=1}^{n} \phi_2 \left\{ 1 - G \left(e^{p_k} (t - \lambda_k) \right) \right\} \right] e^{p_i} (t - \lambda_i) r_Y \left(e^{p_i} (t - \lambda_i) \right)$$

$$\frac{\psi_2 \left[\phi_2 \left\{ 1 - G \left(e^{p_i} (t - \lambda_i) \right) \right\} \right]}{\psi_2' \left[\phi_2 \left\{ 1 - G \left(e^{p_i} (t - \lambda_i) \right) \right\} \right]} \leq 0,$$

proving that $\Psi(\boldsymbol{p})$ is decreasing in each p_i. Moreover, using (7), it can be easily shown that

$$\frac{\partial \Psi}{\partial p_i} - \frac{\partial \Psi}{\partial p_j} \leq (\geq) 0.$$

Thus, by Lemma 2 (Lemma 1), $\Psi(\boldsymbol{p})$ is s-convex in $\boldsymbol{p}$. This proves the result.

To prove ii), using Lemma 4, it is to prove that

$$\psi_2 \left[\sum_{k=1}^{n} \phi_2 \left\{ 1 - G \left(\frac{t - \lambda_k}{\sigma_k} \right) \right\} \right] = \Psi(\boldsymbol{\sigma}),$$

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is increasing in each σ_i and s-convex in $\boldsymbol{\sigma}$. Now, if $\boldsymbol{\lambda}, \boldsymbol{\sigma} \in \mathcal{D}_+ (\mathcal{E}_+)$ and $u^2 r_Y(u)$ is decreasing in u, following similar argument as in the previous result, it can be shown that, for all $i \leq j$,

$$-\left(\frac{t-\lambda_i}{\sigma_i}\right)^2 r_Y\left(\frac{t-\lambda_i}{\sigma_i}\right)\frac{1}{t-\lambda_i}\frac{\psi_2(u_i)}{\psi_2'(u_i)} \geq (\leq) -\left(\frac{t-\lambda_j}{\sigma_j}\right)^2 r_Y\left(\frac{t-\lambda_j}{\sigma_j}\right)\frac{1}{t-\lambda_j}\frac{\psi_2(u_j)}{\psi_2'(u_j)}, \quad (8)$$

with $u_i = \phi_2\left[1 - G\left(\frac{t-\lambda_i}{\sigma_i}\right)\right]$. Now, differentiating $\Psi(\boldsymbol{\sigma})$ with respect to σ_i, we get,

$$\frac{\partial \Psi}{\partial \sigma_i} = \psi_2'\left[\sum_{k=1}^n \phi_2\left\{1 - G\left(\frac{t-\lambda_k}{\sigma_k}\right)\right\}\right]\left(\frac{t-\lambda_i}{\sigma_i}\right)^2 r_Y\left(\frac{t-\lambda_i}{\sigma_i}\right)\frac{1}{t-\lambda_i}$$
$$\frac{\psi_2\left[\phi_2\left(1 - G\left(\frac{t-\lambda_i}{\sigma_i}\right)\right)\right]}{\psi_2'\left[\phi_2\left(1 - G\left(\frac{t-\lambda_i}{\sigma_i}\right)\right)\right]} \geq 0,$$

proving that $\Psi(\boldsymbol{\sigma})$ is increasing in each σ_i. Moreover, using (8), it can be easily shown that

$$\frac{\partial \Psi}{\partial \sigma_i} - \frac{\partial \Psi}{\partial \sigma_j} \geq (\leq)0.$$

Thus, by Lemma 1 (Lemma 2), $\Psi(\boldsymbol{\sigma})$ is s-convex in $\boldsymbol{\sigma}$. □

The next two theorems show that usual stochastic ordering exists between $X_{1:n}$ and $Y_{1:n}$ under majorization order of the location parameters. Proof of the second theorem follows from the first one, and hence is omitted.

Theorem 4 *Let $X_1, X_2, ..., X_n$ be a set of dependent random variables sharing Archimedean copula having generator ψ_1 such that $X_i \sim LS(\lambda_i, \sigma_i, F)$, $i = 1, 2, ..., n$. Let $Y_1, Y_2, ..., Y_n$ be another set of dependent random variables sharing Archimedean copula having generator ψ_2 such that $Y_i \sim LS(\mu_i, \sigma_i, G)$, $i = 1, 2, ..., n$. Assume that $\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\sigma} \in \mathcal{D}_+$ (or $\mathcal{E}_+$). Further suppose that $\phi_2 \circ \psi_1$ is super-additive, ψ_1 or ψ_2 is log-concave and either $ur_X(u)$ or $ur_Y(u)$ is decreasing in u, then, $X \leq_{st} Y$ and $\boldsymbol{\lambda} \preceq_w \boldsymbol{\mu} \Rightarrow X_{1:n} \leq_{st} Y_{1:n}$.*

Proof: Following the same argument as in (5) and (6) of Theorem 3, it can be shown that

$$\psi_1\left[\sum_{k=1}^n \phi_1\left\{1 - F\left(\frac{t-\lambda_k}{\sigma_k}\right)\right\}\right] \leq \psi_2\left[\sum_{k=1}^n \phi_2\left\{1 - G\left(\frac{t-\lambda_k}{\sigma_k}\right)\right\}\right]. \quad (9)$$

As before, to prove the result, it suffices to prove that

$$\psi_2\left[\sum_{k=1}^n \phi_2\left\{1 - G\left(\frac{t-\lambda_k}{\sigma_k}\right)\right\}\right] \leq \psi_2\left[\sum_{k=1}^n \phi_2\left\{1 - G\left(\frac{t-\mu_k}{\sigma_k}\right)\right\}\right].$$

Using Lemma 5, it can be said that the above relation will hold if

$$\Psi(\boldsymbol{\lambda}) = \psi_2\left[\sum_{k=1}^n \phi_2\left\{1 - G\left(\frac{t-\lambda_k}{\sigma_k}\right)\right\}\right]$$

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is increasing in each λ_i and s-convex in $\boldsymbol{\lambda}$. Now, as $\boldsymbol{\lambda}, \boldsymbol{\sigma} \in \mathcal{D}_+$ (or $\mathcal{E}_+$), $u r_Y(u)$ is decreasing in u and ψ_2 is log-concave, following the same argument as of Theorem 3, we can show that

$$-\frac{1}{t-\lambda_i}\left(\frac{t-\lambda_i}{\sigma_i}\right) r_Y\left(\frac{t-\lambda_i}{\sigma_i}\right) \frac{\psi_2(u_i)}{\psi_2'(u_i)} \geq (\leq) -\frac{1}{t-\lambda_j}\left(\frac{t-\lambda_j}{\sigma_j}\right) r_Y\left(\frac{t-\lambda_j}{\sigma_j}\right) \frac{\psi_2(u_j)}{\psi_2'(u_j)}; \ i \leq j,$$

$$(10)$$

with $u_i = \phi_2\left[1 - G\left(\frac{t-\lambda_i}{\sigma_i}\right)\right]$. Differentiating $\Psi(\boldsymbol{\lambda})$ with respect to λ_i, we get

$$\frac{\partial \Psi}{\partial \lambda_i} = \psi_2'\left[\sum_{k=1}^{n} \phi_2\left\{1 - G\left(\frac{t-\lambda_i}{\sigma_i}\right)\right\}\right] \frac{1}{t-\lambda_i} \frac{t-\lambda_i}{\sigma_i} r_Y\left(\frac{t-\lambda_i}{\sigma_i}\right)$$

$$\frac{\psi_2\left[\phi_2\left\{1 - G\left(\frac{t-\lambda_i}{\sigma_i}\right)\right\}\right]}{\psi_2'\left[\phi_2\left\{1 - G\left(\frac{t-\lambda_i}{\sigma_i}\right)\right\}\right]} \geq 0,$$

proving that $\Psi(\boldsymbol{\lambda})$ is increasing in each λ_i. Moreover, using (10), it can be easily shown that

$$\frac{\partial \Psi}{\partial \lambda_i} - \frac{\partial \Psi}{\partial \lambda_j} \geq (\leq)0.$$

Thus, by Lemma 1 (Lemma 2), $\Psi(\boldsymbol{\lambda})$ is s-convex in $\boldsymbol{\lambda}$. This proves the result. $\qquad \square$

Theorem 5 *Let $X_1, X_2, ..., X_n$ be a set of dependent random variables sharing Archimedean copula having generator ψ_1 such that $X_i \sim LS(\lambda_i, \sigma_i, F)$, $i = 1, 2, ..., n$. Let $Y_1, Y_2, ..., Y_n$ be another set of dependent random variables sharing Archimedean copula having generator ψ_2 such that $Y_i \sim LS(\mu_i, \sigma_i, G)$, $i = 1, 2, ..., n$. Assume that $\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\sigma} \in \mathcal{D}_+$ (or $\mathcal{E}_+$). Further suppose that $\phi_2 \circ \psi_1$ is super-additive, ψ_1 or ψ_2 is log-convex and either $r_X(u)$ or $r_Y(u)$ is increasing in u, then, $X \leq_{st} Y$ and $\boldsymbol{\lambda} \overset{w}{\succeq} \boldsymbol{\mu} \Rightarrow X_{1:n} \leq_{st} Y_{1:n}$.*

3.2 Heterogenous independent Components

In this subsection, we will compare two series systems with heterogenous independent LS family distributed components. For $i = 1, 2, \ldots, n$, let X_i and Y_i be two sets of n independent random variables following LS distribution with parameters $(\boldsymbol{\lambda}, \boldsymbol{\sigma})$ and $(\boldsymbol{\mu}, \boldsymbol{\xi})$ respectively, as given in (1). If $\overline{F}_{1:n}(\cdot)$ and $\overline{G}_{1:n}(\cdot)$ be the survival functions of $X_{1:n}$ and $Y_{1:n}$ respectively, then clearly

$$\overline{F}_{1:n}(x) = \prod_{k=1}^{n}\left[1 - F\left(\frac{t-\lambda_k}{\sigma_k}\right)\right]$$

and

$$\overline{G}_{1:n}(x) = \prod_{k=1}^{n}\left[1 - G\left(\frac{t-\mu_k}{\xi_k}\right)\right].$$

Again, if $r_{1:n}(\cdot)$ and $s_{1:n}(\cdot)$ are the hazard rate functions of $X_{1:n}$ and $Y_{1:n}$ respectively then,

$$r_{1:n}(t) = \sum_{k=1}^{n} \frac{1}{\sigma_k} \frac{f\left(\frac{t-\lambda_k}{\sigma_k}\right)}{1 - F\left(\frac{t-\lambda_k}{\sigma_k}\right)} = \sum_{k=1}^{n} \frac{1}{\sigma_k} r_X\left(\frac{t-\lambda_k}{\sigma_k}\right) \qquad (11)$$

and

$$s_{1:n}(t) = \sum_{k=1}^{n} \frac{1}{\xi_k} \frac{g\left(\frac{t-\mu_k}{\xi_k}\right)}{1 - G\left(\frac{t-\mu_k}{\xi_k}\right)} = \sum_{k=1}^{n} \frac{1}{\xi_k} r_Y\left(\frac{t-\mu_k}{\xi_k}\right), \tag{12}$$

In the next two theorems, stochastic comparison between minimum order statistics from the LS family with respect to hazard rate ordering has been discussed. These results strengthen Theorems 1 - 4 for the independent case, as hazard rate ordering implies usual stochastic ordering.

Theorem 6 *Let $X_1, X_2, ..., X_n$ be a set of independent random variables such that $X_i \sim LS(\lambda_i, \sigma_i, F)$, $i = 1, 2, ..., n$. Let $Y_1, Y_2, ..., Y_n$ be another set of independent random variables such that $Y_i \sim LS(\lambda_i, \xi_i, G)$, $i = 1, 2, ..., n$. Suppose that $\boldsymbol{\sigma}, \boldsymbol{\xi}, \boldsymbol{\lambda} \in \mathcal{D}_+$ (or $\mathcal{E}_+$). Then, if $X \leq_{hr} Y$ and*

i) *either $ur_X(u)$ or $ur_Y(u)$ is concave in u, then, $\frac{1}{\boldsymbol{\sigma}} \overset{m}{\preceq} \frac{1}{\boldsymbol{\xi}} \Rightarrow X_{1:n} \leq_{hr} Y_{1:n}$,*

ii) *either $ur_X(u)$ or $ur_Y(u)$ is increasing (decreasing) and concave in u, then, $\frac{1}{\boldsymbol{\sigma}} \overset{w}{\preceq} (\preceq_w)\frac{1}{\boldsymbol{\xi}}$ $\Rightarrow X_{1:n} \leq_{hr} Y_{1:n}$,*

iii) *either $ur_X(u)$ or $ur_Y(u)$ is increasing in u, and $u\frac{d}{du}(ur_X(u))$ or $u\frac{d}{du}(ur_Y(u))$ is decreasing in u, then, $\frac{1}{\boldsymbol{\sigma}} \overset{p}{\preceq} \frac{1}{\boldsymbol{\xi}} \Rightarrow X_{1:n} \leq_{hr} Y_{1:n}$,*

iv) *either $ur_X(u)$ or $ur_Y(u)$ is increasing in u, and $u^2\frac{d}{du}(r_X(u))$ or $u^2\frac{d}{du}(r_Y(u))$ is decreasing in u, then, $\frac{1}{\boldsymbol{\sigma}} \overset{rm}{\preceq} \frac{1}{\boldsymbol{\xi}} \Rightarrow X_{1:n} \leq_{hr} Y_{1:n}$.*

Proof: If $X \leq_{hr} Y$, then $r_X\left(\frac{t-\lambda_k}{\sigma_k}\right) \geq r_Y\left(\frac{t-\lambda_k}{\sigma_k}\right)$, which results in

$$\sum_{k=1}^{n} \frac{r_X\left(\frac{t-\lambda_k}{\sigma_k}\right)}{\sigma_k} \geq \sum_{k=1}^{n} \frac{r_Y\left(\frac{t-\lambda_k}{\sigma_k}\right)}{\sigma_k}.$$

So, to prove the result it suffices to prove that $\sum_{k=1}^{n} \frac{r_Y\left(\frac{t-\lambda_k}{\sigma_k}\right)}{\sigma_k} \geq \sum_{k=1}^{n} \frac{r_Y\left(\frac{t-\lambda_k}{\xi_k}\right)}{\xi_k}$.

Proof of i): Here we have to prove that $\sum_{k=1}^{n} \frac{r_Y\left(\frac{t-\lambda_k}{\sigma_k}\right)}{\sigma_k} = \sum_{k=1}^{n} p_k r_Y(p_k(t-\lambda_k)) = \Psi(\boldsymbol{p})$, is s-concave in $\boldsymbol{p}$. Now, if $\boldsymbol{\lambda}, \boldsymbol{\sigma}, \boldsymbol{\xi} \in \mathcal{D}_+$ (or $\mathcal{E}_+$) then $p_i(t-\lambda_i) \leq (\geq)p_j(t-\lambda_j)$. Thus, the concavity of the function $ur_Y(u)$ gives

$$\frac{d}{du}[ur_Y(u)]_{u=p_i(t-\lambda_i)} - \frac{d}{du}[ur_Y(u)]_{u=p_j(t-\lambda_j)} \geq (\leq)0.$$

Now, differentiating $\Psi(\boldsymbol{p})$ with respect to p_i, we get

$$\frac{\partial \Psi}{\partial p_i} = \frac{d}{du}[ur_Y(u)]_{u=p_i(t-\lambda_i)}, \tag{13}$$

proving that $\frac{\partial \Psi}{\partial p_i} - \frac{\partial \Psi}{\partial p_j} \geq (\leq)0$. Thus, by Lemma 2 (Lemma 1) $\Psi(\boldsymbol{p})$ is s-concave in $\boldsymbol{p}$. This proves the result.

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Proof of ii) : If $ur_Y(u)$ is increasing (decreasing) in u, then from (13) it can be written that that $\Psi(\boldsymbol{p})$ is increasing (decreasing) in p_i. Again, as $ur_Y(u)$ is concave, by the previous result $\Psi(\boldsymbol{p})$ is s-concave in $\boldsymbol{p}$. Thus, by Lemma 5, the result is proved.

Proof of iii) : Let us assume that

$$\sum_{k=1}^{n} \frac{r_Y\left(\frac{t-\lambda_k}{\sigma_k}\right)}{\sigma_k} = \sum_{k=1}^{n} e^{p_k} r_Y\left(e^{p_k}(t-\lambda_k)\right) = \Psi(\boldsymbol{p}),$$

where $\boldsymbol{p} = \left(\log\frac{1}{\sigma_1}, \log\frac{1}{\sigma_2}, ..., \log\frac{1}{\sigma_n}\right)$. So, by Lemma 3, it is suffices to prove that $\Psi(\boldsymbol{p})$ is increasing in each p_i and s-concave in $\boldsymbol{p}$. Now, differentiating $\Psi(\boldsymbol{p})$ with respect to p_i, we get,

$$\begin{aligned}
\frac{\partial \Psi}{\partial p_i} &= e^{p_i}\left[r_Y\left(e^{p_i}(t-\lambda_i)\right)\right] + e^{p_i}(t-\lambda_i)r_Y\left(e^{p_i}(t-\lambda_i)\right) \\
&= e^{p_i}\frac{d}{du}\left[ur_Y(u)\right]_{u=e^{p_i}(t-\lambda_i)} \\
&= \frac{1}{t-\lambda_i}\left[u\frac{d}{du}\left(ur_Y(u)\right)\right]_{u=e^{p_i}(t-\lambda_i)} \\
&> 0.
\end{aligned}$$

The last inequality follows from the fact that $ur_Y(u)$ is increasing in u. Again, as $u\frac{d}{du}(ur_Y(u))$ is decreasing in u, and $\boldsymbol{\lambda}, \boldsymbol{\sigma} \in \mathcal{D}_+(or\mathcal{E}_+)$, then for all $i \le j$,

$$\begin{aligned}
\frac{\partial \Psi}{\partial p_i} - \frac{\partial \Psi}{\partial p_j} &= \frac{1}{t-\lambda_i}\left[u\frac{d}{du}\left(ur_Y(u)\right)\right]_{u=e^{p_i}(t-\lambda_i)} - \frac{1}{t-\lambda_j}\left[u\frac{d}{du}\left(ur_Y(u)\right)\right]_{u=e^{p_j}(t-\lambda_j)} \\
&\ge (\le) \quad 0.
\end{aligned}$$

Hence, the result is followed from Lemma 2 (Lemma 1).

Proof of iv) Let us assume that $\Psi(\boldsymbol{\sigma}) = \sum_{k=1}^{n} \frac{r_Y\left(\frac{t-\lambda_k}{\sigma_k}\right)}{\sigma_k}$. Now, differentiating $\Psi(\boldsymbol{\sigma})$ with respect to σ_i and considering the fact that $ur_Y(u)$ is increasing in u, it can be written that,

$$\begin{aligned}
\frac{\partial \Psi}{\partial \sigma_i} &= \frac{-1}{\sigma_i^2}r_Y\left(\frac{t-\lambda_i}{\sigma_i}\right) - \left(\frac{t-\lambda_i}{\sigma_i^3}\right)r_Y'\left(\frac{t-\lambda_i}{\sigma_i}\right) \\
&= \frac{-1}{(t-\lambda_i)^2}\left(\frac{t-\lambda_i}{\sigma_i}\right)^2 \frac{d}{du}\left[ur_Y(u)\right]_{u=\frac{t-\lambda_i}{\sigma_i}} \\
&\le 0.
\end{aligned}$$

Again, as $\boldsymbol{\lambda}, \boldsymbol{\sigma} \in \mathcal{D}_+(or \ \mathcal{E}_+)$, and $u^2\frac{d}{du}\left[ur_Y(u)\right]$ is decreasing in u, it can be easily shown that $\frac{\partial \Psi}{\partial \sigma_i} - \frac{\partial \Psi}{\partial \sigma_j} \le (\ge)0$. Thus, by Lemma 1 (Lemma 2), $\Psi(\boldsymbol{\sigma})$ is s-concave and hence the result is proved by Lemma 4. $\qquad\square$

Theorem 7 *Let $X_1, X_2, ..., X_n$ be a set of independent random variables such that $X_i \sim LS(\lambda_i, \sigma_i, F)$, $i = 1, 2, ..., n$. Let $Y_1, Y_2, ..., Y_n$ be another set of independent random variables such that $Y_i \sim LS(\lambda_i, \xi_i, G)$, $i = 1, 2, ..., n$. Suppose that $\boldsymbol{\sigma}, \boldsymbol{\xi}, \boldsymbol{\lambda} \in \mathcal{D}_+$ (or $\mathcal{E}_+$). Then, if $X \le_{hr} Y$ and*

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i) *either $ur_X(u)$ or $ur_Y(u)$ is convex in u, then, $\frac{1}{\sigma} \overset{m}{\succeq} \frac{1}{\xi} \Rightarrow X_{1:n} \leq_{hr} Y_{1:n}$,*

ii) *either $ur_X(u)$ or $ur_Y(u)$ is increasing (decreasing) and convex in u, then, $\frac{1}{\sigma} \succeq_w (\overset{w}{\succeq})\frac{1}{\xi} \Rightarrow$ $X_{1:n} \leq_{hr} Y_{1:n}$,*

iii) *either $ur_X(u)$ or $ur_Y(u)$ is decreasing in u, and $u\frac{d}{du}(ur_X(u))$ or $u\frac{d}{du}(ur_Y(u))$ is increasing in u, then, $\frac{1}{\sigma} \overset{p}{\succeq} \frac{1}{\xi} \Rightarrow X_{1:n} \leq_{hr} Y_{1:n}$,*

iv) *either $ur_X(u)$ or $ur_Y(u)$ is decreasing in u, and $u^2\frac{d}{du}(r_X(u))$ or $u^2\frac{d}{du}(r_Y(u))$ is increasing in u, then, $\frac{1}{\sigma} \overset{rm}{\succeq} \frac{1}{\xi} \Rightarrow X_{1:n} \leq_{hr} Y_{1:n}$.*

Proof: Proof of the theorem follows from the previous theorem and hence is omitted. $\square$

In the next theorem, it is shown that under certain restrictions, there exists R-hr ordering between two minimum order statistics obtained from two different LS families, when one set of scale parameters majorizes the other. It is to be mentioned here that, due to mathematical complexities we cannot proceed in full generality, where two LS families could be generated from two different baseline distributions and hence we consider that both LS families are generated from the same baseline distribution.

Theorem 8 *Let $X_1, X_2, ..., X_n$ be a set of independent random variables such that $X_i \sim LS(\lambda_i, \sigma_i, F)$, $i = 1, 2, ..., n$. Let $Y_1, Y_2, ..., Y_n$ be another set of independent random variables such that $Y_i \sim LS(\lambda_i, \xi_i, F)$, $i = 1, 2, ..., n$. Suppose that $\boldsymbol{\lambda}, \boldsymbol{\sigma}, \boldsymbol{\xi} \in \mathcal{D}_+$ (or $\mathcal{E}_+$). If $r_X(u)$ is r-concave for $r = 1, 2$ and 3 then, $\frac{1}{\sigma} \overset{m}{\preceq} \frac{1}{\xi} \Rightarrow X_{1:n} \leq_{R-hr} Y_{1:n}$.*

Proof: To prove the result we have only to prove that $g(t) = \frac{s_{1:n}(t)}{r_{1:n}(t)}$ is decreasing in t. Now,

$$g'(t) = \frac{d}{dt}\left[\frac{\sum_{k=1}^{n} \frac{1}{\xi_k} r_X\left(\frac{t-\lambda_k}{\xi_k}\right)}{\sum_{k=1}^{n} \frac{1}{\sigma_k} r_X\left(\frac{t-\lambda_k}{\sigma_k}\right)}\right] \overset{sign}{=} \frac{\sum_{k=1}^{n} \frac{1}{\xi_k^2} r_X'\left(\frac{t-\lambda_k}{\xi_k}\right)}{\sum_{k=1}^{n} \frac{1}{\xi_k} r_X'\left(\frac{t-\lambda_k}{\xi_k}\right)} - \frac{\sum_{k=1}^{n} \frac{1}{\sigma_k^2} r_X'\left(\frac{t-\lambda_k}{\sigma_k}\right)}{\sum_{k=1}^{n} \frac{1}{\sigma_k} r_X'\left(\frac{t-\lambda_k}{\sigma_k}\right)}.$$

So, it is to be proved that

$$\frac{\sum_{k=1}^{n} \frac{r_X'\left(\frac{t-\lambda_k}{\sigma_k}\right)}{\sigma_k^2}}{\sum_{k=1}^{n} \frac{r_X'\left(\frac{t-\lambda_k}{\sigma_k}\right)}{\sigma_k}} = \frac{\sum_{k=1}^{n} p_k^2 r_X'(p_k(t-\lambda_k))}{\sum_{k=1}^{n} p_k r_X(p_k(t-\lambda_k))} = \frac{\sum_{k=1}^{n} u(p_k,\lambda_k,t)v(p_k,\lambda_k,t)}{\sum_{k=1}^{n} u(p_k,\lambda_k,t)} = \frac{\sum_{k=1}^{n} u_k v_k}{\sum_{k=1}^{n} u_k} = \Psi(\boldsymbol{p}) \ (say),$$

with $\boldsymbol{p} = \left(\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, ..., \frac{1}{\sigma_n}\right)$, is s-concave in $\boldsymbol{p}$, where $u(p_k, \lambda_k, t) = p_k r_X(p_k(t - \lambda_k)) = u_k$ (say) and $v(p_k, \lambda_k, t) = p_k \frac{r_X'(p_k(t-\lambda_k))}{r_X(p_k(t-\lambda_k))} = v_k$ (say). Now, differentiating $\Psi(\boldsymbol{p})$ with respect to p_i we get,

$$\frac{\partial \Psi}{\partial p_i} \overset{sign}{=} \sum_{k=1}^{n} u_k\left(v_i\frac{\partial u_i}{\partial p_i} + u_i\frac{\partial v_i}{\partial p_i}\right) - \sum_{k=1}^{n} u_k v_k\left(\frac{\partial u_i}{\partial p_i}\right). \tag{14}$$

Now, differentiating u_k and v_k with respect to p_k, it can be written that
$\frac{\partial u_k}{\partial p_k} = u_k w_k$; $w_k = \frac{1 + v_k(t-\lambda_k)}{p_k}$, and

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$$\frac{\partial v_k}{\partial p_k} = v_k \left[\frac{1}{p_k} + (t - \lambda_k)\frac{r_X''(p_k(t-\lambda_k))}{r_X'(p_k(t-\lambda_k))} - (t - \lambda_k)\frac{r_X'(p_k(t-\lambda_k))}{r_Y(p_k(t-\lambda_k))} \right] = v_k s_k \text{ (say)},$$

where $s_k = \frac{1}{p_k} + (t - \lambda_k)\frac{r_X''(p_k(t-\lambda_k))}{r_X'(p_k(t-\lambda_k))} - (t - \lambda_k)\frac{r_X'(p_k(t-\lambda_k))}{r_Y(p_k(t-\lambda_k))}$.

Using (14) and expressions for $\frac{\partial u_k}{\partial p_k}$ and $\frac{\partial v_k}{\partial p_k}$, it can be shown that,

$$\frac{\partial \Psi}{\partial p_i} \overset{sign}{=} \left(\sum_{k=1}^n u_k \right)(u_i v_i w_i + u_i v_i s_i) - \left(\sum_{k=1}^n u_k \right) v_k u_i w_i,$$

which gives, for all $i \le j$

$$\frac{\partial \Psi}{\partial p_i} - \frac{\partial \Psi}{\partial p_j} \overset{sign}{=} \left(\sum_{k=1}^n u_k \right)[u_i v_i(w_i + s_i) - u_j v_j(w_j + s_j)] - \left(\sum_{k=1}^n u_k v_k \right)(u_i w_i - u_j w_j), \quad (15)$$

with

$$\begin{aligned}
u_i v_i(w_i + s_i) - u_j v_j(w_j + s_j) &= 2 \left[p_i r_X'(p_i(t - \lambda_i)) - p_j r_X'(p_j(t - \lambda_j)) \right] \\
&\quad + \left[p_i^2(t - \lambda_i)r_X''(p_i(t - \lambda_i)) - p_j^2 r_X''(p_j(t - \lambda_j)) \right].
\end{aligned} \quad (16)$$

Now as $\boldsymbol{\lambda}, \boldsymbol{\sigma} \in \mathcal{D}_+, (or\ \mathcal{E}_+)$, then for all $i \le j$, $p_i, < p_j$, which gives $p_i(t - \lambda_i) \le (\ge)p_j(t - \lambda_j)$ and $p_i^2(t - \lambda_i) \le (\ge)p_j^2(t - \lambda_j)$. Moreover, as $r_X(u)$ is s-concave for $s = 1, 2$, it is decreasing and concave and hence it can be concluded that $p_i r_X'(p_i(t - \lambda_i)) \ge (\le)p_j r_X'(p_j(t - \lambda_j))$. So, the first term of (16) is positive (negative).

Again, as $r_X(u)$ is also 3-concave and thus $r_X''(u)$ decreasing in u, then for all $i \le j$ it can be written that, $r_X''(p_i(t - \lambda_i)) \ge (\le)r_X''(p_j(t - \lambda_j))$, which gives $p_i^2(t - \lambda_i)r_X''(p_i(t - \lambda_i)) \ge (\le)p_j^2(t - \lambda_j)r_X''(p_j(t - \lambda_j))$. So the second term of (16) is also positive (negative). Thus the first term of (15) is positive (negative).

Now, following the arguments as before it can be easily shown that,

$$u_i w_i - u_j w_j = [r_X(p_i(t - \lambda_i)) - r_X(p_j(t - \lambda_j))] + \left[p_i(t - \lambda_i)r_X'(p_i(t - \lambda_i)) - p_j(t - \lambda_j)r_X'(p_j(t - \lambda_j)) \right] \ge (\le)0.$$

Now as $r_X(u)$ is decreasing in u, $v_k = p_k \frac{r_X'(p_k(t-\lambda_k))}{r_X(p_k(t-\lambda_k))}$ is negative. So, the second term of (15) is also positive (negative). Hence, $\frac{\partial \Psi}{\partial p_i} - \frac{\partial \Psi}{\partial p_j} \ge (\le)0$. Thus by Lemma 2 (Lemma 1) it can be concluded that $\Psi(\boldsymbol{p})$ is s-concave in $\boldsymbol{p}$. This proves the result. $\square$

Systems are also compared stochastically in situations where the components are from multiple-outlier models. This is to be mentioned here that, a multiple-outlier model is a set of independent random variables $X_1, ..., X_n$ of which $X_i \overset{st}{=} X, i = 1, ..., n_1$ and $X_i \overset{st}{=} Y, i = n_1 + 1, ..., n$ where $1 \le n_1 < n$ and $X_i \overset{st}{=} X$ means that cdf of X_i is same as that of X. In other words, the set of independent random variables $X_1, ..., X_n$ is said to constitute a multiple-outlier model if two sets of random variables $(X_1, X_2, \ldots, X_{n_1})$ and $(X_{n_1+1}, X_{n_1+2}, \ldots, X_{n_1+n_2})$ (where $n_1 + n_2 = n$), are homogenous among themselves and heterogenous between themselves. For more details on multiple-outlier models, readers may refer to Kochar and Xu ([29]), Zhao and

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Balakrishnan ([30]), Balakrishnan and Torrado ([31]), Zhao and Zhang [32], Kundu *et al.* [8], Kundu and Chowdhury [9] and the references there in. Two theorems given below show that, under certain conditions, majorized scale parameter vectors of LS distributed components of series system for multiple outlier model, leads to smaller system life in terms of R-hr ordering.

Theorem 9 *Let* $\{X_1, X_2, \ldots, X_n\}$ *and* $\{Y_1, Y_2, \ldots, Y_n\}$ *be two sets of independent random variables each following the multiple outlier LS model such that* $X_i \sim LS(\lambda_1, \sigma_1, F)$ *and* $Y_i \sim LS(\lambda_1, \xi_1, F)$ *for* $i = 1, 2, \ldots, n_1$ *and* $X_i \sim LS(\lambda_2, \sigma_2, F)$ *and* $Y_i \sim LS(\lambda_2, \xi_2, F)$ *for* $i = n_1 + 1, n_1 + 2, \ldots, n_1 + n_2 = n$. *Suppose that* $(\lambda_1, \lambda_2), (\sigma_1, \sigma_2), (\xi_1, \xi_2) \in \mathcal{D}_+$ *(or* $\mathcal{E}_+$*). Now if* $u r_X(u)$ *is decreasing in* u *and* $r_X(u)$ *is log-concave and 2-log-convex in* u, *then*

$$(\underbrace{\frac{1}{\sigma_1}, \frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_1}}_{n_1}, \underbrace{\frac{1}{\sigma_2}, \frac{1}{\sigma_2}, \ldots, \frac{1}{\sigma_2}}_{n_2}) \overset{m}{\preceq} (\underbrace{\frac{1}{\xi_1}, \frac{1}{\xi_1}, \ldots, \frac{1}{\xi_1}}_{n_1}, \underbrace{\frac{1}{\xi_2}, \frac{1}{\xi_2}, \ldots, \frac{1}{\xi_2}}_{n_2}) \Rightarrow X_{1:n} \geq_{R-hr} Y_{1:n}.$$

Proof: To prove the result we have only to prove that $g(t) = \frac{s_{1:n}(t)}{r_{1:n}(t)}$ is increasing in t. Now,

$$g'(t) = \frac{d}{dt}\left[\frac{\sum_{k=1}^{n} \frac{1}{\xi_k} r_X\left(\frac{t-\lambda_k}{\xi_k}\right)}{\sum_{k=1}^{n} \frac{1}{\sigma_k} r_X\left(\frac{t-\lambda_k}{\sigma_k}\right)}\right] \overset{sign}{=} \frac{\sum_{k=1}^{n} \frac{1}{\xi_k^2} r_X'\left(\frac{t-\lambda_k}{\xi_k}\right)}{\sum_{k=1}^{n} \frac{1}{\xi_k} r_X'\left(\frac{t-\lambda_k}{\xi_k}\right)} - \frac{\sum_{k=1}^{n} \frac{1}{\sigma_k^2} r_X'\left(\frac{t-\lambda_k}{\sigma_k}\right)}{\sum_{k=1}^{n} \frac{1}{\sigma_k} r_X'\left(\frac{t-\lambda_k}{\sigma_k}\right)}.$$

So, to prove the result it is to be shown that

$$\frac{\sum_{k=1}^{n} \frac{r_X'\left(\frac{t-\lambda_k}{\sigma_k}\right)}{\sigma_k^2}}{\sum_{k=1}^{n} \frac{r_X'\left(\frac{t-\lambda_k}{\sigma_k}\right)}{\sigma_k}} = \frac{\sum_{k=1}^{n} p_k^2 r_X'\left(p_k(t-\lambda_k)\right)}{\sum_{k=1}^{n} p_k r_X\left(p_k(t-\lambda_k)\right)} = \frac{\sum_{k=1}^{n} u(p_k, \lambda_k, t) v(p_k, \lambda_k, t)}{\sum_{k=1}^{n} u(p_k, \lambda_k, t)} = \frac{\sum_{k=1}^{n} u_k v_k}{\sum_{k=1}^{n} u_k} = \Psi(\boldsymbol{p}) \ (say),$$

with $\boldsymbol{p} = \left(\underbrace{p_1, \ldots, p_1}_{n_1}, \underbrace{p_2, \ldots, p_2}_{n_2}\right) = \left(\underbrace{\frac{1}{\sigma_1}, \frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_1}}_{n_1}, \underbrace{\frac{1}{\sigma_2}, \frac{1}{\sigma_2} \ldots \frac{1}{\sigma_2}}_{n_2}\right)$, is s-convex in $\boldsymbol{p}$, where

$u(p_k, \lambda_k, t) = p_k r_X\left(p_k(t-\lambda_k)\right) = u_k$ (say) and $v(p_k, \lambda_k, t) = p_k \frac{r_X'(p_k(t-\lambda_k))}{r_X(p_k(t-\lambda_k))} = v_k$ (say).

Let $i \leq j$. Now three cases may arise:

Case(i): If $1 \leq i, j \leq n_1$, i.e., if $p_i = p_j = p_1$ and $\lambda_i = \lambda_j = \lambda_1$, then

$$\frac{\partial \Psi}{\partial p_i} - \frac{\partial \Psi}{\partial p_j} = \frac{\partial \Psi}{\partial p_1} - \frac{\partial \Psi}{\partial p_1} = 0.$$

Case(ii): If $n_1 + 1 \leq i, j \leq n$, i.e., if $p_i = p_j = p_2$ and $\lambda_i = \lambda_j = \lambda_2$, then

$$\frac{\partial \Psi}{\partial p_i} - \frac{\partial \Psi}{\partial p_j} = \frac{\partial \Psi}{\partial p_2} - \frac{\partial \Psi}{\partial p_2} = 0.$$

Case(iii): If $1 \leq i \leq n_1$ and $n_1 + 1 \leq j \leq n$, then $p_i = p_1$, $p_j = p_2$, $\lambda_i = \lambda_1$ and $\lambda_j = \lambda_2$. Then,

$$\frac{\partial \Psi}{\partial p_i} \left(n_1 u_1 + n_2 u_2\right)^2 = n_2 u_2 \frac{\partial u_1}{\partial p_1} \left(v_1 - v_2\right) + w_1 \left(n_1 u_1 + n_2 u_2\right),$$

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and

$$\frac{\partial \Psi}{\partial p_j} (n_1 u_1 + n_2 u_2)^2 = n_1 u_1 \frac{\partial u_2}{\partial p_2} (v_2 - v_1) + w_2 (n_1 u_1 + n_2 u_2),$$

where, for $i = 1, 2$, $w_i = u_i \frac{\partial v_i}{\partial p_i}$. Thus

$$\frac{\partial \Psi}{\partial p_i} - \frac{\partial \Psi}{\partial p_j} \overset{sign}{=} (v_1 - v_2) \left(n_2 u_2 \frac{\partial u_1}{\partial p_1} + n_1 u_1 \frac{\partial u_2}{\partial p_2} \right) + (w_1 - w_2)(n_1 u_1 + n_2 u_2). \quad (17)$$

Now, as $u r_X(u)$ is decreasing in u, then for $i = 1, 2$, $\frac{\partial u_i}{\partial p_i} = \frac{d}{du}(u r_X(u))|_{u=(t-\lambda_i)p_i} < 0$. Now, if $(\lambda_1, \lambda_2), (\sigma_1, \sigma_2) \in \mathcal{D}_+$ (or $\mathcal{E}_+$), then $p_1 \leq (\geq)p_2$ and $p_1(t - \lambda_1) \leq (\geq)p_2(t - \lambda_2)$. Thus, as $r_X(u)$ is log-concave in u, by considering the fact that $r_X(u)$ is decreasing in u, it can be written that

$$-p_1 \frac{r_X'(p_1(t - \lambda_1))}{r_X(p_1(t - \lambda_1))} \leq (\geq) - p_2 \frac{r_X'(p_2(t - \lambda_2))}{r_X(p_2(t - \lambda_2))},$$

giving that $v_1 \geq (\leq)v_2$. So, the first term of (17) is negative (positive). Again, for $i = 1, 2$,

$$\begin{aligned} w_i &= p_i r_X(t - \lambda_i) \frac{\partial}{\partial p_i} \left(\frac{r_X'(p_i(t - \lambda_i))}{r_X(p_i(t - \lambda_i))} \right) \\ &= p_i(t - \lambda_i) r_X(t - \lambda_i) \frac{\partial^2}{\partial u^2} (\log r_X(u))|_{u=p_i(t-\lambda_i)}. \end{aligned}$$

So, considering the fact that $r_X(u)$ is log-concave giving $\frac{\partial^2}{\partial u^2} \log r_X(u) < 0$, and $r_X(u)$ is also 2-log-convex, it can be written that,

$$-p_1(t - \lambda_1) r_X(t - \lambda_1) \frac{\partial^2}{\partial u^2} (\log r_X(u))|_{u=p_1(t-\lambda_1)} \geq (\leq) - p_2(t - \lambda_2) r_X(t - \lambda_2) \frac{\partial^2}{\partial u^2} (\log r_X(u))|_{u=p_2(t-\lambda_2)},$$

yielding that $w_1 \leq (\geq)w_2$. So, the second term of (17) is also negative (positive). Thus, as $\frac{\partial \Psi}{\partial p_i} - \frac{\partial \Psi}{\partial p_j} \leq (\geq)0$, by Lemma 2 (Lemma 1) it can be concluded that $\Psi(\boldsymbol{p})$ is s-convex in $\boldsymbol{p}$. This proves the result. $\qquad \square$

The next theorem can be proved in the similar line as of previous theorem.

Theorem 10 *Let $\{X_1, X_2, \ldots, X_n\}$ and $\{Y_1, Y_2, \ldots, Y_n\}$ be two sets of independent random variables each following the multiple outlier LS model such that $X_i \sim LS(\lambda_1, \sigma_1, F)$ and $Y_i \sim LS(\lambda_1, \xi_1, F)$ for $i = 1, 2, \ldots, n_1$ and $X_i \sim LS(\lambda_2, \sigma_2, F)$ and $Y_i \sim LS(\lambda_2, \xi_2, F)$ for $i = n_1 + 1, n_1 + 2, \ldots, n_1 + n_2 = n$. Suppose that $(\lambda_1, \lambda_2), (\sigma_1, \sigma_2), (\xi_1, \xi_2) \in \mathcal{D}_+$ (or $\mathcal{E}_+$). Now if $u r_X(u)$ is increasing in u and $r_X(u)$ is decreasing, log-concave and 2-log-concave in u, then*

$$(\underbrace{\frac{1}{\sigma_1}, \frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_1}}_{n_1}, \underbrace{\frac{1}{\sigma_2}, \frac{1}{\sigma_2}, \ldots, \frac{1}{\sigma_2}}_{n_2}) \overset{m}{\succeq} (\underbrace{\frac{1}{\xi_1}, \frac{1}{\xi_1}, \ldots, \frac{1}{\xi_1}}_{n_1}, \underbrace{\frac{1}{\xi_2}, \frac{1}{\xi_2}, \ldots, \frac{1}{\xi_2}}_{n_2}) \Rightarrow X_{1:n} \geq_{R-hr} Y_{1:n}.$$

Now, combining Theorem 6 i) and Theorem 9 it can be shown that, for multiple outlier LS model, majorized scale parameter vector of minimum order statistics leads to smaller system life in terms of lr ordering. The statement of the theorem is given below.

Theorem 11 *Let $\{X_1, X_2, \ldots, X_n\}$ and $\{Y_1, Y_2, \ldots, Y_n\}$ be two sets of independent random variables each following the multiple outlier LS model such that $X_i \sim LS(\lambda_1, \sigma_1, F)$ and $Y_i \sim LS(\lambda_1, \xi_1, F)$ for $i = 1, 2, \ldots, n_1$ and $X_i \sim LS(\lambda_2, \sigma_2, F)$ and $Y_i \sim LS(\lambda_2, \xi_2, F)$ for $i = n_1 + 1, n_1 + 2, \ldots, n_1 + n_2 = n$. Suppose that $(\lambda_1, \lambda_2), (\sigma_1, \sigma_2), (\xi_1, \xi_2) \in \mathcal{D}_+$ (or $\mathcal{E}_+$). Now if $ur_X(u)$ is decreasing and concave in u and $r_X(u)$ is log-concave and 2-log-convex in u, then*

$$(\underbrace{\frac{1}{\sigma_1}, \frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_1}}_{n_1}, \underbrace{\frac{1}{\sigma_2}, \frac{1}{\sigma_2}, \ldots, \frac{1}{\sigma_2}}_{n_2}) \overset{m}{\preceq} (\underbrace{\frac{1}{\xi_1}, \frac{1}{\xi_1}, \ldots, \frac{1}{\xi_1}}_{n_1}, \underbrace{\frac{1}{\xi_2}, \frac{1}{\xi_2}, \ldots, \frac{1}{\xi_2}}_{n_2}) \Rightarrow X_{1:n} \leq_{lr} Y_{1:n}.$$

Again combining Theorem 7 i) and Theorem 10 the following theorem can be obtained.

Theorem 12 *Let $\{X_1, X_2, \ldots, X_n\}$ and $\{Y_1, Y_2, \ldots, Y_n\}$ be two sets of independent random variables each following the multiple outlier LS model such that $X_i \sim LS(\lambda_1, \sigma_1, F)$ and $Y_i \sim LS(\lambda_1, \xi_1, F)$ for $i = 1, 2, \ldots, n_1$ and $X_i \sim LS(\lambda_2, \sigma_2, F)$ and $Y_i \sim LS(\lambda_2, \xi_2, F)$ for $i = n_1 + 1, n_1 + 2, \ldots, n_1 + n_2 = n$. Suppose that $(\lambda_1, \lambda_2), (\sigma_1, \sigma_2), (\xi_1, \xi_2) \in \mathcal{D}_+$ (or $\mathcal{E}_+$). Now if $ur_X(u)$ is increasing and convex in u and $r_X(u)$ is decreasing, log-concave and 2-log-convex in u, then*

$$(\underbrace{\frac{1}{\sigma_1}, \frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_1}}_{n_1}, \underbrace{\frac{1}{\sigma_2}, \frac{1}{\sigma_2}, \ldots, \frac{1}{\sigma_2}}_{n_2}) \overset{m}{\succeq} (\underbrace{\frac{1}{\xi_1}, \frac{1}{\xi_1}, \ldots, \frac{1}{\xi_1}}_{n_1}, \underbrace{\frac{1}{\xi_2}, \frac{1}{\xi_2}, \ldots, \frac{1}{\xi_2}}_{n_2}) \Rightarrow X_{1:n} \leq_{lr} Y_{1:n}.$$

Now the question arises $-$ what will happen if location parameter vector of one LS family majorizes the other when the scale parameter vector remains same? The next few theorems deal with such cases.

The next theorem shows that under some restrictions, majorization ordering between location parameter vectors of minimum order statistics from two different LS family of distributions implies hr ordering between them.

Theorem 13 *Let $\{X_1, X_2, \ldots, X_n\}$ and $\{Y_1, Y_2, \ldots, Y_n\}$ be two sets of independent random variables each following LS model such that $X_i \sim LS(\lambda_i, \sigma_i, F)$ and $Y_i \sim LS(\mu_i, \sigma_i, G)$ for $i = 1, 2, \ldots, n$. Suppose further that $\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\sigma} \in \mathcal{D}_+$ (or $\mathcal{E}_+$). Then, if $X \leq_{hr} Y$, either $r_X(u)$ or $r_Y(u)$ is increasing (decreasing) in u and either $u^2 r_X'(u)$ or $u^2 r_Y'(u)$ is decreasing (increasing) in u then, $\boldsymbol{\lambda} \preceq_w (\succeq_w)\boldsymbol{\mu} \Rightarrow X_{1:n} \leq_{hr} Y_{1:n}$.*

Proof: If $X \leq_{hr} Y$, then $r_X\left(\frac{t - \lambda_k}{\sigma_k}\right) \geq r_Y\left(\frac{t - \lambda_k}{\sigma_k}\right)$, which results in

$$\sum_{k=1}^n \frac{r_X\left(\frac{t - \lambda_k}{\sigma_k}\right)}{\sigma_k} \geq \sum_{k=1}^n \frac{r_Y\left(\frac{t - \lambda_k}{\sigma_k}\right)}{\sigma_k}.$$

So, to prove the result it suffices to prove that $\sum_{k=1}^n \frac{r_Y\left(\frac{t - \lambda_k}{\sigma_k}\right)}{\sigma_k} \geq \sum_{k=1}^n \frac{r_Y\left(\frac{t - \mu_k}{\sigma_k}\right)}{\sigma_k}$, or equivalently it is required to show that $\sum_{k=1}^n \frac{r_Y\left(\frac{t - \mu_k}{\sigma_k}\right)}{\sigma_k} = \Psi(\boldsymbol{\mu})$ is decreasing (increasing) in each μ_i and s-concave (s-convex) in $\boldsymbol{\mu}$. Now, differentiating Ψ with respect to μ_i it can be written that $\frac{\partial \Psi}{\partial \mu_i} =$

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$-\frac{1}{\sigma_i^2}r_Y'\left(\frac{t-\mu_i}{\sigma_i}\right)$, which is clearly decreasing (increasing) in μ_i if $r_Y(u)$ is increasing (decreasing) in u. Again, if $\boldsymbol{\mu}, \boldsymbol{\sigma} \in \mathcal{D}_+$, then for all $i \leq j$, $\mu_i \geq \mu_j$, $\sigma_i \geq \sigma_j$ giving $\frac{t-\mu_i}{\sigma_i} \leq \frac{t-\mu_j}{\sigma_j}$. So, if $r_Y(u)$ is increasing (decreasing) and $u^2 r_Y'(u)$ is decreasing (increasing) in u, then

$$\frac{1}{(t-\mu_i)^2}\left(\frac{t-\mu_i}{\sigma_i}\right)^2 r_Y'\left(\frac{t-\mu_i}{\sigma_i}\right) \geq (\leq)\frac{1}{(t-\mu_j)^2}\left(\frac{t-\mu_j}{\sigma_j}\right)^2 r_Y'\left(\frac{t-\mu_j}{\sigma_j}\right)$$

giving that

$$\frac{\partial \Psi}{\partial \mu_i} - \frac{\partial \Psi}{\partial \mu_j} = \frac{1}{\sigma_j^2}r_Y'\left(\frac{t-\mu_j}{\sigma_j}\right) - \frac{1}{\sigma_i^2}r_Y'\left(\frac{t-\mu_i}{\sigma_i}\right) \leq (\geq)0.$$

So, by Lemma 1 it can be concluded that $\Psi(\boldsymbol{\mu})$ is s-concave (s-convex) in $\boldsymbol{\mu}$. This proves the result.

When $\boldsymbol{\mu}, \boldsymbol{\sigma} \in \mathcal{E}_+$, then the theorem can be proved in similar line. $\qquad \square$

The theorem given below shows that for two LS families of distribution functions generated from the same baseline distribution, there exists R-hr ordering between their corresponding minimum order statistics.

Theorem 14 *Let $\{X_1, X_2, \ldots, X_n\}$ and $\{Y_1, Y_2, \ldots, Y_n\}$ be two sets of independent random variables each following LS model such that $X_i \sim LS(\lambda_i, \sigma_i, F)$ and $Y_i \sim LS(\mu_i, \sigma_i, F)$ for $i = 1, 2, \ldots, n$. Suppose further that $\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\sigma} \in \mathcal{D}_+$ (or $\mathcal{E}_+$). Then, if $r_X(u)$ is increasing and concave (decreasing and convex) in u; $u^2 r_X'(u)$ is decreasing (increasing) in u and $u^3 r_X''(u)$ is increasing (decreasing) in u then, $\boldsymbol{\lambda} \preceq_w (\succeq_w)\boldsymbol{\mu} \Rightarrow X_{1:n} \geq_{R-hr} Y_{1:n}$.*

Proof: To prove the result we have only to prove that $g(t) = \frac{s_{1:n}(t)}{r_{1:n}(t)}$ is increasing in t. Now,

$$g'(t) = \frac{d}{dt}\left[\frac{\sum_{k=1}^{n}\frac{1}{\sigma_k}r_X\left(\frac{t-\mu_k}{\sigma_k}\right)}{\sum_{k=1}^{n}\frac{1}{\sigma_k}r_X\left(\frac{t-\lambda_k}{\sigma_k}\right)}\right] \overset{sign}{=} \frac{\sum_{k=1}^{n}\frac{1}{\sigma_k^2}r_X'\left(\frac{t-\mu_k}{\sigma_k}\right)}{\sum_{k=1}^{n}\frac{1}{\xi_k}r_X'\left(\frac{t-\mu_k}{\sigma_k}\right)} - \frac{\sum_{k=1}^{n}\frac{1}{\sigma_k^2}r_X'\left(\frac{t-\lambda_k}{\sigma_k}\right)}{\sum_{k=1}^{n}\frac{1}{\sigma_k}r_X'\left(\frac{t-\lambda_k}{\sigma_k}\right)}.$$

So, it is to be proved that

$$\frac{\sum_{k=1}^{n}\frac{r_X'\left(\frac{t-\mu_k}{\sigma_k}\right)}{\sigma_k^2}}{\sum_{k=1}^{n}\frac{r_X'\left(\frac{t-\mu_k}{\sigma_k}\right)}{\sigma_k}} = \Psi(\boldsymbol{\mu})$$

is increasing (decreasing) in each μ_i and s-convex (s-concave) in $\boldsymbol{\mu}$. Now differentiating Ψ with respect to μ_i,

$$\frac{\partial \Psi}{\partial \mu_i}\left[\sum_{k=1}^{n}\frac{1}{\sigma_k}r_X\left(\frac{t-\mu_k}{\sigma_k}\right)\right]^2 = -\left[\sum_{k=1}^{n}\frac{1}{\sigma_k}r_X\left(\frac{t-\mu_k}{\sigma_k}\right)\right]\frac{1}{\sigma_i^3}r_X''\left(\frac{t-\mu_i}{\sigma_i}\right) + \left[\sum_{k=1}^{n}\frac{1}{\sigma_k^2}r_X'\left(\frac{t-\mu_k}{\sigma_k}\right)\right]\frac{1}{\sigma_i^2}r_X'\left(\frac{t-\mu_i}{\sigma_i}\right)$$

which is increasing (decreasing) in each μ_i if $r_X(u)$ is increasing and concave (decreasing and convex) in u. Again,

$$\begin{aligned}\frac{\partial \Psi}{\partial \mu_i} - \frac{\partial \Psi}{\partial \mu_j} &\overset{sign}{=} \left[\sum_{k=1}^{n}\frac{1}{\sigma_k}r_X\left(\frac{t-\mu_k}{\sigma_k}\right)\right]\left[\frac{1}{\sigma_j^3}r_X''\left(\frac{t-\mu_j}{\sigma_j}\right) - \frac{1}{\sigma_i^3}r_X''\left(\frac{t-\mu_i}{\sigma_i}\right)\right] \\ &\quad + \left[\sum_{k=1}^{n}\frac{1}{\sigma_k^2}r_X'\left(\frac{t-\mu_k}{\sigma_k}\right)\right]\left[\frac{1}{\sigma_i^2}r_X'\left(\frac{t-\mu_i}{\sigma_i}\right) - \frac{1}{\sigma_j^2}r_X'\left(\frac{t-\mu_j}{\sigma_j}\right)\right].\end{aligned} \tag{18}$$

Now if $\boldsymbol{\mu}, \boldsymbol{\sigma} \in \mathcal{D}_+$, then for all $i \leq j$, $t - \mu_i \leq t - \mu_j$, $\sigma_i \geq \sigma_j$ and $\frac{t-\mu_i}{\sigma_i} \leq \frac{t-\mu_j}{\sigma_j}$. So, as $u^2 r'_X(u)$ is decreasing (increasing) in u then for all $i \leq j$,

$$\left(\frac{t-\mu_i}{\sigma_i}\right)^2 r'_X\left(\frac{t-\mu_i}{\sigma_i}\right) \geq (\leq) \left(\frac{t-\mu_j}{\sigma_j}\right)^2 r'_X\left(\frac{t-\mu_j}{\sigma_j}\right).$$

As, $r_X(u)$ is increasing (decreasing) in u, then from the above relation it can be written that

$$\frac{1}{(t-\mu_i)^2}\left(\frac{t-\mu_i}{\sigma_i}\right)^2 r'_X\left(\frac{t-\mu_i}{\sigma_i}\right) \geq (\leq) \frac{1}{(t-\mu_j)^2}\left(\frac{t-\mu_j}{\sigma_j}\right)^2 r'_X\left(\frac{t-\mu_j}{\sigma_j}\right).$$

So, the second term of (18) is positive (negative). Again, $u^3 r''_X(u)$ is increasing (decreasing) in u gives

$$\left(\frac{t-\mu_i}{\sigma_i}\right)^3 r''_X\left(\frac{t-\mu_i}{\sigma_i}\right) \leq (\geq) \left(\frac{t-\mu_j}{\sigma_j}\right)^3 r''_X\left(\frac{t-\mu_j}{\sigma_j}\right).$$

Considering the fact that $r_X(u)$ is concave (convex) in u, from the above relation it can be written that,

$$\frac{1}{(t-\mu_i)^3}\left(\frac{t-\mu_i}{\sigma_i}\right)^3 r''_X\left(\frac{t-\mu_i}{\sigma_i}\right) \geq (\leq) \frac{1}{(t-\mu_j)^3}\left(\frac{t-\mu_j}{\sigma_j}\right)^3 r''_X\left(\frac{t-\mu_j}{\sigma_j}\right).$$

So, the first term of (18) is also positive (negative). Thus $\frac{\partial \Psi}{\partial \mu_i} - \frac{\partial \Psi}{\partial \mu_j} \geq (\leq) 0$. So by Lemma 1 it can be concluded that Ψ is s-convex (s-concave) in $\boldsymbol{\mu}$. Thus, using Lemma 5 the required result can be obtained.

When $\boldsymbol{\mu}, \boldsymbol{\sigma} \in \mathcal{E}_+$, then the theorem can be proved in similar line. $\qquad\square$

Combining Theorem 13 and Theorem 14 the following theorem can easily be obtained.

Theorem 15 *Let $\{X_1, X_2, \ldots, X_n\}$ and $\{Y_1, Y_2, \ldots, Y_n\}$ be two sets of independent random variables each following LS model such that $X_i \sim LS(\lambda_i, \sigma_i, F)$ and $Y_i \sim LS(\mu_i, \sigma_i, F)$ for $i = 1, 2, \ldots, n$. Suppose further that $\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\sigma} \in \mathcal{D}_+$ (or $\mathcal{E}_+$). Then, if $r_X(u)$ is increasing and concave (decreasing and convex) in u; $u^2 r'_X(u)$ is decreasing (increasing) in u and $u^3 r''_X(u)$ is increasing (decreasing) in u then, $\boldsymbol{\lambda} \preceq_w (\succeq_w) \boldsymbol{\mu} \Rightarrow X_{1:n} \leq_{lr} Y_{1:n}$.*

4   COMPARISON OF PARALLEL SYSTEMS WITH LS DISTRIBUTED COMPONENTS

Similar results for the parallel system are furnished in this section for dependent setup. These results are the extension of the Theorems 3.1-3.5 of Hazra *et al.* [12], where the parallel system consists of independent LS family distributed components. The following theorems 16-18 can be proved in the same line as of Theorem 1, Theorem 3 and Theorem 4 respectively and hence the proofs are omitted. For results of independent random variables, one may refer to Hazra *et al.* [12].

Let X and Y be two random variables having distribution functions $F(\cdot)$ and $G(\cdot)$ respectively. Suppose that $X_i \sim LS(\lambda_i, \sigma_i, F)$ and $Y_i \sim LS(\mu_i, \xi_i, G)$ $(i = 1, 2, \ldots, n)$ be two sets of n

dependent random variables with Archimedean copulas having generators ψ_1 (with $\phi_1 = \psi_1^{-1}$) and ψ_2 ($\phi_2 = \psi_2^{-1}$) respectively. Also suppose that $G_{n:n}(\cdot)$ and $H_{n:n}(\cdot)$ be the distribution functions of $X_{n:n}$ and $Y_{n:n}$ respectively. Then,

$$G_{n:n}(t) = \psi_1 \left[\sum_{k=1}^{n} \phi_1 \left\{ F\left(\frac{t - \lambda_k}{\sigma_k} \right) \right\} \right], \ t > max(\lambda_k, \ \forall k),$$

and

$$H_{n:n}(t) = \psi_2 \left[\sum_{k=1}^{n} \phi_2 \left\{ G\left(\frac{t - \mu_k}{\xi_k} \right) \right\} \right], \ t > max(\mu_k, \ \forall k).$$

Let $\tilde{r}_X(u)$ and $\tilde{r}_Y(u)$ are the reversed hazard rate functions of the random variables X and Y respectively.

Theorem 16 *Let $X_1, X_2, ..., X_n$ be a set of dependent random variables sharing Archimedean copula having generator ψ_1 such that $X_i \sim LS(\lambda, \sigma_i, F)$, $i = 1, 2, ..., n$. Let $Y_1, Y_2, ..., Y_n$ be another set of dependent random variables sharing Archimedean copula having generator ψ_2 such that $Y_i \sim LS(\lambda, \xi_i, G)$, $i = 1, 2, ..., n$. Assume that $\boldsymbol{\lambda}, \boldsymbol{\sigma}, \boldsymbol{\xi} \in \mathcal{D}_+$ (or $\mathcal{E}_+$). Further suppose that $\phi_2 \circ \psi_1$ is super-additive, ψ_1 or ψ_2 is log-convex and $X \geq_{st} Y$. If either $\tilde{r}_X(u)$ or $\tilde{r}_Y(u)$ is decreasing in u then, $\frac{1}{\boldsymbol{\sigma}} \overset{w}{\succeq} \frac{1}{\boldsymbol{\xi}} \Rightarrow X_{n:n} \geq_{st} Y_{n:n}$.*

Theorem 17 *Let $X_1, X_2, ..., X_n$ be a set of dependent random variables sharing Archimedean copula having generator ψ_1 such that $X_i \sim LS(\lambda_i, \sigma_i, F)$, $i = 1, 2, ..., n$. Let $Y_1, Y_2, ..., Y_n$ be another set of dependent random variables sharing Archimedean copula having generator ψ_2 such that $Y_i \sim LS(\lambda_i, \xi_i, G)$, $i = 1, 2, ..., n$. Assume that $\boldsymbol{\lambda}, \boldsymbol{\sigma}, \boldsymbol{\xi} \in \mathcal{D}_+$ (or $\mathcal{E}_+$). Further suppose that $\phi_2 \circ \psi_1$ is super-additive, ψ_1 or ψ_2 is log-convex and $X \geq_{st} Y$, then,*

i) $\frac{1}{\boldsymbol{\sigma}} \overset{p}{\succeq} \frac{1}{\boldsymbol{\xi}} \Rightarrow X_{n:n} \geq_{st} Y_{n:n}$, if either $ur_X(u)$ or $ur_Y(u)$ is decreasing in u;

ii) $\frac{1}{\boldsymbol{\sigma}} \overset{rm}{\succeq} \frac{1}{\boldsymbol{\xi}} \Rightarrow X_{n:n} \geq_{st} Y_{n:n}$, if either $u^2 r_X(u)$ or $u^2 r_Y(u)$ is decreasing in u.

Theorem 18 *Let $X_1, X_2, ..., X_n$ be a set of dependent random variables sharing Archimedean copula having generator ψ_1 such that $X_i \sim LS(\lambda_i, \sigma_i, F)$, $i = 1, 2, ..., n$. Let $Y_1, Y_2, ..., Y_n$ be another set of dependent random variables sharing Archimedean copula having generator ψ_2 such that $Y_i \sim LS(\mu_i, \sigma_i, G)$, $i = 1, 2, ..., n$. Assume that $\boldsymbol{\lambda}, \boldsymbol{\sigma}, \boldsymbol{\mu} \in \mathcal{D}_+$ (or $\mathcal{E}_+$). Further suppose that $\phi_2 \circ \psi_1$ is super-additive, ψ_1 or ψ_2 is log-convex and either $ur_X(u)$ or $ur_Y(u)$ is decreasing in u, then, $X \geq_{st} Y$ and $\boldsymbol{\lambda} \succeq_w \boldsymbol{\mu} \Rightarrow X_{n:n} \geq_{st} Y_{n:n}$.*

References

[1] H.A. David and H.N. Nagaraja, Order Statistics, 3rd edition, Wiley, New Jersey, 2003.

[2] R. Dykstra, S.C. Kochar and J. Rojo, "Stochastic comparisons of parallel systems of heterogeneous exponential components," *Journal of Statistical Planning and Inference*, vol. 65, pp. 203-211, 1997.

[3] P. Zhao and N. Balakrishnan, "New results on comparison of parallel systems with heterogeneous gamma components," *Statistics and Probability Letters*, vol. 81, pp. 36-44, 2011.

[4] N. Balakrishnan, G. Barmalzan and A. Haidari, "On usual multivariate stochastic ordering of order statistics from heterogeneous beta variables," *Journal of Multivariate Analysis*, vol. 127, pp. 147-150, 2014.

[5] N. Torrado and S.C. Kochar, "Stochastic order relations among parallel systems from Weibull distributions," *Journal of Applied Probability*, vol. 52, pp. 102-116, 2015.

[6] N. Torrado, "On magnitude orderings between smallest order statistics from heterogeneous beta distributions," *Journal of Mathematical Analysis and Applications*, **426**, pp. 824-838, 2015.

[7] L. Fang and N. Balakrishnan, "Ordering results for the smallest and largest order statistics from independent heterogeneous exponential−Weibull random variables," *Statistics*, vol. 50 (6), pp. 1195-1205, 2016.

[8] A. Kundu, S. Chowdhury, A.K. Nanda and N. Hazra, "Some Results on Majorization and Their Applications," *Journal of Computational and Applied Mathematics*, vol. 301, pp. 161-177, 2016.

[9] A. Kundu and S. Chowdhury, "Ordering properties of order statistics from heterogeneous exponentiated Weibull models," *Statistics and Probability Letters*, vol. 114, pp. 119-127, 2016.

[10] A. Kundu and S. Chowdhury, "Ordering properties of sample minimum from Kumaraswamy-G random variables," *Statistics*, http://dx.doi.org/10.1080/02331888.2017.1353516, 2017.

[11] S. Chowdhury and A. Kundu, "Stochastic Comparison of Parallel Systems with Log-Lindley Distributed Components," *Operations Research Letters*, vol. 45 (3), pp. 199-205.

[12] N.K. Hazra, M.R. Kuiti, M. Finkelstein and A.K. Nanda, "On stochastic comparisons of maximum order statistics from the location-scale family of distributions," *Journal of Multivariate Analysis*, vol. 160, pp. 31-41, 2017.

[13] D. Sengupta and J. V. Deshpande, "Some results on the relative ageing of two life distributions," *Journal of Applied Probability*, vol. 31, pp. 991-1003, 1994.

[14] M. Rezaei, B. Gholizadeh, and S. Izadkhah, "On relative reversed hazard rate order," *Communications in Statistics-Theory and Methods*, vol. 44, pp. 300-308, 2015.

[15] C. Li, and X. Li, "Relative ageing of series and parallel systems with statistically independent and heterogeneous component lifetimes," *IEEE Transactions on Reliability*, vol. 65, pp. 1014-1021, 2016.

[16] J. Navarro and F. Spizzichino, "Comparisons of series and parallel systems with components sharing the same copula," *Applied Stochastic Models in Business and Industry*, vol. 26 (6), pp. 775-791, 2010.

[17] M. Rezapour and M.H. Alamatsaz, "Stochastic comparison of lifetimes of two (n-k+1)-out-of-n systems with heterogeneous dependent components," *Journal of Multivariate Analysis*, vol. 130, pp. 240-251, 2014.

[18] C. Li, and X. Li, "Likelihood ratio order of sample minimum from heterogeneous Weibull random variables," *Statistics and Probability Letters*, vol. 97, pp. 46-53, 2015.

[19] X. Li, and R. Fang, "Ordering properties of order statistics from random variables of Archimedean copulas with applications," *Journal of Multivariate Analysis*, vol. 133, pp. 304-320, 2015.

[20] C. Li, R. Fang, and X. Li, "Stochastic comparisons of order statistics from scaled and interdependent random variables," *Metrika*, vol. 79 (5), pp. 553-578, 2015.

[21] R. Fang, C. Li, and X. Li, "Stochastic comparisons on sample extremes of dependent and heterogenous observations," *Statistics*, vol. 50, pp. 930-955, 2016.

[22] B.E. Khaledi, S. Farsinezhad, and S.C. Kochar, "Stochastic comparisons of order statistics in the scale model," *Journal of Statistical Planning and Inference*, vol. 141, pp. 276-286, 2011.

[23] S.C. Kochar, and N. Torrado, On Stochastic comparisons of largest order statistics in the scale model, *Communications in Statistics-Theory and Methods*, DOI:10.1080/03610926.2014.985839, 2015.

[24] A.W. Marshall, I. Olkin and B.C. Arnold, Inequalities: Theory of Majorization and Its Applications, Springer series in Statistics, New York, 2011.

[25] B.E. Khaledi, S.C. Kochar, "Dispersive ordering among linear combinations of uniform random variables," Journal of Statistical Planning and Inference, vol. 100, pp. 1321, 2002.

[26] M. Shaked, M. and J.G. Shanthikumar, Stochastic Orders, Springer, New York, 2007.

[27] R.B. Nelsen, An Introduction to Copulas, Springer: New York, 2006.

[28] A.J. McNeil and J. Něslehová, "Multivariate Archimedean copulas, D-monotone functions and l1-norm symmetric distributions," *Annals of Statistics*, vol. 37, pp. 3059-3097, 2009.

[29] S. Kochar, and M. Xu, On the skewness of order statistics in multiple-outlier models. *Journal of Applied Probability*, vol. 48, pp. 271-284, 2011.

[30] P. Zhao, and N. Balakrishnan, Stochastic comparison of largest order statistics from multiple-outlier exponential models. *Probability in the Engineering and Informational Sciences*, vol. 26, pp. 159-182, 2012.

[31] N. Balakrishnan, and N. Torrado, Comparisons between largest order statistics from multiple-outlier models. *Statistics*, vol. 50.1, pp. 176-189, 2016.

[32] P. Zhao, and Y. Zhang, On Sample Ranges in Multiple-Outlier Models. *Journal of Multivariate Analysis*, vol. 111, pp. 335-349, 2012.

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